Elements of Conditional Optimization and their Applications to Order Theory

DISSERTATION

zur Erlangung des akademischen Grades

doctor rerum naturalium
(Dr. Rer. Nat.)
im Fach Mathematik

eingereicht an der
Mathematisch-Naturwissenschaftlichen Fakultät II
der Humboldt-Universität zu Berlin

von
Herrn Dipl.-Math. Martin Karliczek

Präsident der der Humboldt-Universität zu Berlin:
Prof. Dr. Jan-Hendrik Olbertz

Dekan der Mathematisch-Naturwissenschaftlichen Fakultät II:
Prof. Dr. Elmar Kulke

Gutachter:
1. Prof. Dr. Michael Kupper
2. Prof. Dr. Dirk Becherer
3. Prof. Dr. Igor Cialenco
4. Prof. Dr. Andreas H. Hamel

eingereicht am: 28.03.2014
Tag der mündlichen Prüfung: 28.11.2014

Für meine Familie.

Abstract

In this thesis, we prove results relevant for optimization problems in L^0-modules and study applications to order theory.

The first part deals with the notion of an Assessment Index (AI), which is used to assess (to evaluate) the own financial position. For an L^0-module $\mathcal{X}$ an AI is a quasiconcave, monotone and local function mapping to L^0. This is a generalization of both real-valued AIs and risk measures mapping to L^0. We first prove a robust representation of these AIs. Then we introduce the concept of dynamic AIs together with notions of consistency and show different representations for them, in particular in the case of the past evolution of the underlying portfolio influences the current assessment.

In the second chapter of this thesis, we develop Ekeland's variational principle for L^0-modules allowing for an L^0-metric. We define local premetrics and prove an L^0-Version of a generalization of Ekeland's theorem. Moreover, we prove L^0-versions of the Kirk-Caristi fixed point theorem and Takahashi's minimization theorem and show the connection of the different results.

A further application of L^0-theory is examined in the third chapter of this thesis, namely an extension of the Brouwer fixed point theorem to functions on $(L^0)^d$. We define a conditional simplex, which is a simplex with respect to L^0, and prove that every local, sequentially continuous function has a fixed point. With this result at hand, we extend the fixed point theorem to arbitrary closed, L^0-convex sets.

A more general structure than L^0-modules is the concept of conditional sets. In the fourth chapter of the thesis, we study conditional topological vector spaces. We examine the concept of duality for conditional sets and prove results of functional analysis: among others, the Banach-Alaoglu and the Krein-Šmulian theorem. Any L^0-module being a conditional set allows to apply all results to L^0-theory.

In the fifth chapter, we discuss the property of transitivity of relations and its connection to certain forms of representations. After a survey of common representations of preferences, we attend to relations induced by moving convex sets which are relations of the form that x is preferred to y if and only if $x-y$ is in a convex set depending on y. We examine in which cases such a representation is transitive. Finally, we exhibit nontransitivity due to dissimilarity of the compared object and discuss representations for relations of that type.

Zusammenfassung

In dieser Arbeit beweisen wir für Optimierungsprobleme in L^0-Moduln relevante Resultate und untersuchen Anwendungen für die Darstellung von Präferenzen.

Im ersten Kapitel der Dissertation geht es um Assessment Indizes (AIs), die genutzt werden, um die eigene Position auf dem Finanzmarkt zu bewerten. Ein AI ist eine quasikonkave, monotone und lokale Funktion von einem L^0-Modul $\mathcal{X}$ nach L^0. Ein AI dieser Form ist sowohl allgemeiner als reellwertige AIs als auch als Risikomaße, die nach L^0 abbilden. Wir stellen diese AIs robust dar. Danach definieren wir dynamische AIs zusammen mit deren Konsistenzbegriffen. Wir beweisen verschiedene robuste Darstellungen dynamischer AIs, unter anderem der Form, dass der Einfluss der Vergangenheit sichtbar ist.

Im zweiten Kapitel entwickeln wir das Ekeland'sche Variationsprinzip für L^0-Moduln, die eine L^0-Metrik besitzen. Wir definieren lokale Prämetriken auf allgemeinen Räumen und beweisen eine L^0-Variante einer Verallgemeinerung des Ekeland'schen Theorems. Darüberhinaus beweisen wir L^0-Versionen des Kirk-Caristi Fixpunktsatzes und des Takahashi Minimization Theorem und zeigen den Zusammenhang zwischen den verschiedenen Resultaten.

Eine weitere Anwendung der L^0-Theorie, nämlich der Beweis des Brouwerschen Fixpunktsatzes für Funktionen, die auf $(L^0)^d$ definiert sind, wird in Kapitel 3 behandelt. Wir definieren das Konzept des Simplexes in $(L^0)^d$ und beweisen zunächst, dass jede lokale, folgenstetige Funktion darauf einen Fixpunkt besitzt. Dies nutzen wir, um den Fixpunktsatz auch für Funktionen auf beliebigen abgeschlossenen, L^0-konvexen Mengen zu zeigen.

Eine weitaus allgemeinere Struktur als L^0 ist die sogenannte bedingte Menge. Im vierten Kapitel der Dissertation behandeln wir bedingte topologische Vektorräume. Wir führen das Konzept der Dualität für bedingte Mengen ein und beweisen Theoreme der Funktionalanalysis darauf, unter anderem das Theorem von Banach-Alaoglu und Krein-Šmulian. Da L^0-Moduln spezielle bedingte Mengen sind, können wir die Resultate dieses Kapitels für die L^0-Theorie nutzen.

Im fünften Kapitel geht es um die Eigenschaft der Transitivität von Relationen und wie diese sich in Darstellungen von Präferenzen niederschlägt. Nachdem gängige Darstellungsformen erklärt werden, widmen wir uns der Darstellung mit wandernden konvexen Mengen. Dabei besitzt jedes Element y eine konvexe Menge $C(y)$, die dessen präferierte Elemente darstellt, in der Art, dass x genau dann besser als y ist, wenn $x - y$ in $C(y)$ liegt. Wir zeigen danach, wie die Transitivität für diese Darstellungsform beschrieben werden kann. Abschließend modellieren wir die Eigenschaft, dass die Transitivität einer Relation nur für ähnliche Elemente gesichert ist und diskutieren Arten der Darstellung solcher Relationen.

Acknowledgements

I hereby would like to express my deepest gratitude towards my supervisor Michael Kupper for his guidance, support and continuous encouragement during my time as a Ph.D. student.

I deeply thank Samuel Drapeau for his creativity, IT consulting and support with the different working projects.

I also owe special thanks to Andreas Hamel for the organization of my research stay in New York and for the fruitful discussions about convex analysis, optimization and many further topics.

I am very grateful to every co-author of our papers, apart from the previously mentioned Tomasz Bielecki, Igor Cialenco, Asgar Jamneshan, Martin Streckfuß, for many helpful discussions and driving the projects. I further thank Christoph Mainberger for the nice time as Ph.D. students and the useful hints for improving my language skills. My years of study were very pleasant due to the nice atmosphere in the community of Humboldt-Universität created by the students mentioned above as well as Julio Backhoff, Martin Büttner, Victor Feunou, Gregor Heyne, Klébert Kentia, Ludovic Tangpi and many others. I would also like to thank Dirk Becherer for his kindness to be available as co-examiner.

Part of my research was done while receiving the funding of the Konsul Karl und Dr. Gabriele Sandmann Stiftung Berlin. I gratefully acknowledge their financial support which enabled me to concentrate on my research.

Contents

Introduction

The main subject of this thesis is the study of L^0-modules and the transfer of objects and theorems of optimization theory to this framework. In addition, we examine representations of preferences in both a stochastic and a deterministic framework.

On a probability space $(\Omega, \mathcal{F}, P)$ we consider the set of all random variables, which are identified if they coincide almost surely. This set is denoted by $L^0(\Omega, \mathcal{F}, P)$ or L^0 for short. It allows for addition and multiplication, both with neutral elements, and moreover provides an inverse with respect to addition. This makes L^0 to be an algebraic ring. However, it does not guarantee an inverse with respect to multiplication which causes it not to be an algebraic field. Hence, one cannot define L^0-vector spaces but L^0-module. The approach of considering L^0 as a substitute for the real numbers has first been done by Cheridito et al. [24], Filipović et al. [43] and Guo [57]. Therein, it turned out that the two main technical properties to demand when working in this setting are locality of functions and σ-stability of sets. In L^0-modules, the action of indicator functions of partitions on sets and functions is essential. In the first three chapters of this thesis, we examine three different problems within the theory of L^0-modules. To do so, we permanently pay attention to the properties of locality and σ-stability. Thus, one challenge of this part of the thesis was to define concepts as the left-inverse of functions and premetrics or simplexes which are local or σ-stable, respectively. A generalization of L^0, namely conditional sets, was introduced by Drapeau et al. [31]. In the fourth chapter of this thesis, we provide a theory of topological vector spaces in that setting. The main focus is to develop a duality theory for conditional sets. Since L^0-modules are specific kinds of conditional sets, all results of Chapter 3 applied to them.

Another key issue of this thesis is the representation of preferences. This is included in the first and fifth chapter. In the fifth chapter, we study general relations with main focus on (non)transitivity. Moreover, we consider representations which are described locally, mainly by convex sets. In the first chapter, the preference is connected to randomness. Using techniques from L^0-theory, we provide a robust representation of utility functions mapping to L^0. To this end, we define the concept of a left-inverse for these functions and have to develop methods of convex analysis in the framework of L^0-modules. Following the ideas of Chapter 1 and using the techniques of Chapter 4, one could study preferences with values in a conditional set.

In the following, we give a more detailed survey of the different chapters of the thesis. The first chapter provides a study of Assessment Indices (AIs) in a discrete time

dynamic framework. It is based on the paper [10] by Bielecki, Cialenco, Drapeau and Karliczek. Assessment indices can be used to assess (to evaluate) various risks, but they can also be used to assess the trade-off between risks and corresponding rewards. Thus, the universe of AIs encompasses both the classical risk measures and the acceptability indices. Consequently, the two basic operational paradigms, that underlie the mathematical theory of assessment indices, are well appreciated truths in any kind of economic/financial activities:
(A) Diversification is better than concentration;
(B) Greater success is better than lesser success.

These two stylized key paradigms translate mathematically into quasiconcavity and monotonicity properties of an AI. In the static case, these two properties were studied in the context of preferences in Cerreia-Vioglio et al. [18, 19] and Drapeau and Kupper [30]. The numerical representations corresponding to preference orderings satisfying properties (A) and (B) cover, among others, risk measures (compare Artzner et al. [4], Föllmer and Schied [48], Frittelli and Rosazza Gianin [52]), as well as acceptability indices (compare Cherny and Madan [25]). In the first chapter of the thesis, we significantly extend previous studies regarding assessment indices to the conditional/dynamic setting, which, in particular, allows us to apply our theory to study dynamic AIs acting on discrete time stochastic processes. The main mathematical tool which we use here in order to derive extension results of Drapeau and Kupper [30] to the conditional setting, is the theory of L^0-modules that was originated in Filipović et al. [43] and in Kupper and Vogelpoth [61]. Similar extension problems have also been studied in Frittelli and Maggis [50, 51], Bion-Nadal [14], Biagini and Bion-Nadal [9]. Here, we provide a study in the general setting of locally convex topological L^0-modules inspired by the methods and techniques of [30]. In many ways, the present work continues and builds upon research of other people that has been presented in numerous works. For obvious reasons we can't provide here the comprehensive list of all these works. Besides the papers that we have already mentioned above, we think that the following works should be brought to the reader's attention: for dynamic, translation-invariant risk measures for processes (with dual representation via probability measures and discount processes) compare Acciaio et al. [2], Cheridito and Kupper [20] and Cheridito et al. [23]; for dynamic, translation-invariant risk measures for random variables (with dual representation via probability measures) compare Bielecki et al. [11], Cheridito et al. [21, 22] and Frittelli and Scandolo [54]; for static assessment indices compare Cherny and Madan [26] and Frittelli and Rosazza Gianin [53]; for conditional risk measures on random variables compare Detlefsen and Scandolo [28]; for dynamic conditional assessment indices compare Biagini and Bion-Nadal [9] and Bion-Nadal [13, 14]; for dual representations for the static quasiconvex case compare Cerreia-Vioglio et al. [18, 19] and Penot and Volle [66]; for conditional quasiconvex functionals on random variables compare Frittelli and Maggis [50, 51].

In the second chapter of this thesis, we develop an Ekeland's variational principle for L^0-modules which allow for an L^0-metric. In 1974 Ekeland introduced the variation principle in [39]. This was applied to optimization problems of real-valued functions. Moreover, it turned out that the principle is equivalent to several other results, for instance the Kirk-Caristi fixed point theorem. Hamel [59] proved a generalization of Ekeland's principle determining which properties are really necessary to prove it. We transfer this generalization to L^0-modules, prove the Kirk-Caristi fixed point theorem and Takahashi's minimization theorem for L^0-modules and show the connection between the individual results. Throughout this chapter, we show how concepts for elements can be translated to sets. This allows to obtain results for set-valued maps on L^0-modules. The chapter is based on a current project of Hamel and Karliczek, however there the results will be proven in a slightly more general setting. To be in line with the other chapters, we decided to present the results for L^0-modules.

The third chapter deals with the Brouwer fixed point theorem in $(L^0)^d$ and corresponds to the paper [32] by Drapeau, Karliczek, Kupper and Streckfuß. The Brouwer fixed point theorem states that a continuous function from a compact and convex set in $\mathbb{R}^d$ to itself has a fixed point. This result and its extensions play a central role in analysis, optimization and economic theory among others. One approach to show the result is to consider functions on simplexes first and use Sperner's lemma. Recently, Cheridito et al. [24], inspired by the theory developed by Filipović et al. [43] and Guo [57], studied $(L^0)^d$ as an L^0-module, discussing concepts like linear independence, σ-stability, locality and L^0-convexity. Based on this, we define affine independence and conditional simplexes in $(L^0)^d$. Starting with a conditional simplex, we can construct a sequence of conditional simplexes converging to an element. We show that this element is a fixed point which is measurable by construction. Hence, even though we follow the constructions and methods used in the proof of the classical result in $\mathbb{R}^d$ (compare Border [15]), we do not need any measurable selection argument. In probabilistic analysis theory the problem of finding random fixed points of random operators is an important issue. Given $\mathcal{C}$, a compact convex subset of a Banach space, a continuous random operator is a function $R\colon \Omega \times \mathcal{C} \to \mathcal{C}$ satisfying

(i) $R(.,x)\colon \Omega \to \mathcal{C}$ is a random variable for any fixed $x \in \mathcal{C}$,

(ii) $R(\omega,.)\colon \mathcal{C} \to \mathcal{C}$ is a continuous function for any fixed $\omega \in \Omega$.

For R there exists a random fixed point which is a random variable $\xi\colon \Omega \to \mathcal{C}$ such that $\xi(\omega) = R(\omega, \xi(\omega))$ for any ω (compare Bharucha-Reid [8], Fierro et al. [42] and Shahzad [67]). In contrast to this ω-wise consideration, our approach is completely within the theory of L^0. All objects and properties are therefore defined in that language and proofs are done using L^0-methods. Moreover, the connection between continuous random operators on $\mathbb{R}^d$ and sequentially continuous functions on $(L^0)^d$ is not entirely clear.

In the fourth chapter of the thesis, we study conditional topological vector spaces. The concept of a conditional set was introduced by Drapeau et al. [31]. The idea of a conditional set is to consider a set which is determined under various conditions. These conditions are modeled as a Boolean Algebra and its influence as an action of it. In the conditional set theory one defines objects and properties such that they still can be investigated under the different conditions. To do mathematics with these objects the properties of consistency and stability are requested. These properties clarify how an element of the conditional set behaves if the condition under which this element is examined is extended or restricted. The definition of stability is closely related to the property of σ-stability in L^0-modules. The set $L^0 = L^0(\Omega, \mathcal{F}, P)$ can be seen as a conditional set where the action of conditioning is the restriction of a random variable to an element in $\mathcal{F}$. The difference is that in conditional set theory one has objects available which only appear on certain conditions but not on the whole set Ω. Specifically, in L^0 random variables are defined on the whole Ω whereas the conditional set associated to L^0 also contains elements which are random variables only defined on A, with $A \in \mathcal{F}$. The advantage of considering L^0 as a conditional set is the usefulness of the conditional power set which makes topological properties to be nicely handled. Based on conditional set theory, we examine the concept of duality for conditional sets. To this end, we introduce the notion of a locally convex topological vector space in this setting. Then, we define structures like norm, dual pair and polar cone, and prove among others the Bipolar, Banach-Alaoglu and Krein-Šmulian Theorem.

In the fifth chapter, we discuss the representation of relations with emphasis on the property of transitivity. We give a survey of the different forms of representing relations and their connection to transitivity. The common way of representing a relation is to define a global criteria. For example, one determines a set U of functions from $\mathcal{X}$ to $\mathbb{R}$ such that x is better then y, with $x, y \in \mathcal{X}$, if and only if $u(x) \geq u(y)$ for every $u \in U$. The set U is hence applied independently of x and y and the representation is called multi-utility representation. Our approach is different as we include a dependence of U on the element we want to compare with. This means there is a set $U(y)$ such that x is better then y, if and only if $u(x) \geq u(y)$ for every $u \in U(y)$. However, if we want to determine the elements which are better then x we use the function set $U(x)$. In this way we have a local description of the relation. In the second part of this chapter, we analyze representations induced by moving sets. This means an element x is better than an element y if and only if $x - y$ is in a set $C(y)$ depending on y. If $C(y) = C$ is a constant set for every y, this is well-known. We aim at a description of transitivity in terms of the $C(y)$. We show that this is possible for closed, convex cones and give examples why it does not work anymore if we drop one of these properties. Furthermore, we are looking at the connection to a multi-utility representation. Finally, we explain how one can relax the property of transitivity of a relation. Transitivity means that in chains such that x is better than y and y is better than z it always holds that also x is better than z. We consider the case where transitivity fails, since elements become too dissimilar in long chains of comparison. By an example, we illustrate how this behavior appears in real life. We then model this kind of transitivity mathematically and give several types of representations for relations fulfilling it.

Introduction to L^0-theory

The first three chapters of the thesis deal with L^0-modules. We will give an overview of the basic concepts and notions for the work in L^0-modules and will mainly follow the notation of Filipović et al. [43].

Let $(\Omega, \mathcal{F}, P)$ be a probability space. By $L^0 = L^0(\Omega, \mathcal{F}, P)$ and $\overline{L^0}$ we denote the set of all $\mathcal{F}$-measurable random variables, P-almost surely identified, with values in $\mathbb{R}$ and $\overline{\mathbb{R}}$, respectively. In $\mathcal{F}$ we identify sets A, B for which $P((A \backslash B) \cup (B \backslash A))=0$. For $\mathcal{G} \subseteq \mathcal{F}$, we denote by $\vee \mathcal{G}$ the supremum and by $\wedge \mathcal{G}$ the infimum of $\mathcal{G}$, respectively, with respect to set-inclusion. By $\mathcal{F}_+$ we denote the set of all events $A \in \mathcal{F}$ with $P(A) > 0$.

We will use the notation $\geq$ for the P-almost sure greater than or equal to relation and define the sets $L^0_+ = \{X \in L^0 : X \geq 0\}$ and $L^0_{++} = \{X \in L^0 : X > 0\}$. Moreover, for a subset $\mathcal{M}$ of L^0, we denote by $\operatorname{ess\,sup} \mathcal{M}$ the essential supremum of it, which is the element $Y \in L^0$ such that $Y \geq X$ for all $X \in \mathcal{M}$ and for all $Z \in L^0$ which also fulfill $Z \geq X$ for all $X \in \mathcal{M}$ it always holds $Z \geq Y$. The essential supremum always exists in $\overline{L^0}$ (compare Filipović et al. [43]). We work with the convention that $0 \cdot \infty = 0$. In this way, for example $1_A \infty \in \overline{L^0}$ is well-defined, where 1_A denotes the indicator function of $A \in \mathcal{F}$. A partition of Ω is a countable family of sets $(A_n)_{n \in \mathbb{N}} \subseteq \mathcal{F}$ such that $\bigcup_{n \in \mathbb{N}} A_n = \Omega$ and $A_n \cap A_m = \emptyset$ if $n \neq m, n, m \in \mathbb{N}$.

An L^0-module $\mathcal{X}$ is an abelian group equipped with a scalar multiplication $L^0 \times \mathcal{X} \to \mathcal{X}, (\lambda, X) \mapsto \lambda X$ fulfilling

- $\lambda(\mu X) = (\lambda\mu)X$,
- $(\lambda + \mu)X = \lambda X + \mu X$,
- $\lambda(X + Y) = \lambda X + \lambda Y$,

for all $X, Y \in \mathcal{X}$ and $\lambda, \mu \in L^0$.

We will only consider L^0-modules which are σ-stable meaning that for every partition $(A_n)_{n \in \mathbb{N}}$ and any sequence $(X_n)_{n \in \mathbb{N}}$ the object $\sum_{n \in \mathbb{N}} 1_{A_n} X_n$ is a unique element in $\mathcal{X}$ such that $1_{A_n} \sum_{n \in \mathbb{N}} 1_{A_n} X_n = 1_{A_n} X_n$ for every $n \in \mathbb{N}$. In particular it holds that $\sum_{n \in \mathbb{N}} 1_{A_n} X = X$ for any $X \in \mathcal{X}$. The concept of σ-stability was introduced by Filipović et al. [43] and turned out to be the main tool for the work with L^0-modules. Given a σ-stable set $\mathcal{X}$, the σ-stable hull of $\mathcal{C} \subseteq \mathcal{X}$ is defined as

$$\sigma(\mathcal{C}) = \left\{ \sum_{i \in \mathbb{N}} 1_{A_i} X_i : X_i \in \mathcal{C}, (A_i)_{i \in \mathbb{N}} \text{ is a partition} \right\}.$$

We call a nonempty set $\mathcal{C}$ σ-stable if it is equal to $\sigma(\mathcal{C})$. For two σ-stable sets $\mathcal{X}$ and $\mathcal{Y}$ a function $f : \mathcal{X} \to \mathcal{Y}$ is called local if for any partition $(A_n)_{n\in\mathbb{N}}$ and any sequence $(X_n)_{n\in\mathbb{N}}$ it holds $f(\sum_{n\in\mathbb{N}} 1_{A_n} X_n) = \sum_{n\in\mathbb{N}} 1_{A_n} f(X_n)$. In the case of two L^0-modules $\mathcal{X}$ and $\mathcal{Y}$, σ-stability in $\mathcal{X} \times \mathcal{Y}$ reads as: for every partition $(A_n)_{n\in\mathbb{N}}$ and elements $(X_n)_{n\in\mathbb{N}} \subseteq \mathcal{X}$, $(Y_n)_{n\in\mathbb{N}} \subseteq \mathcal{Y}$ it follows $\sum_{n\in\mathbb{N}} 1_{A_n}(X_n, Y_n) = (\sum_{n\in\mathbb{N}} 1_{A_n} X_n, \sum_{n\in\mathbb{N}} 1_{A_n} Y_n) \in \mathcal{X} \times \mathcal{Y}$.

We denote by $\mathbb{N}(\mathcal{F}) = \{X \in L^0 : P(X \in \mathbb{N}) = 1\}$ the set of all random variables mapping to $\mathbb{N}$. Elements of $\mathbb{N}(\mathcal{F})$ are typically denoted by N. A local sequence in $\mathcal{X}$ is a local function from $\mathbb{N}(\mathcal{F})$ to $\mathcal{X}$, $N \mapsto X_N$ and is denoted by $(X_N)_{N\in\mathbb{N}(\mathcal{F})}$ or just (X_N). We will briefly recall concepts of L^0-theory at the beginning of Chapters 1-3.

1 Dynamic Assessment Indices

This part deals with the notion of Assessment Indices (AI) and corresponds to the paper [10] by Bielecki, Cialenco, Drapeau and Karliczek. An AI is used to evaluate one's own position in portfolio optimization. For an L^0-module $\mathcal{X}$ an AI is a quasiconcave, monotone and local function mapping to L^0. This is on the one hand more general than real-valued AI and on the other hand even more general than risk measures mapping to L^0. We give a robust representation of these AI. Afterwards we introduce the concept of a dynamic AI, define the notions of consistency and represent them, with special focus on a representation where the influence of the past is still visible. The main motivation for our study of Assessment Indices in a dynamic setup, henceforth called Dynamic Assessment Indices (DAI), is the analysis of risks and rewards propagating in time. In contrast to the static case, the study of DAIs bears additional conceptual difficulties related to the conditionality and to the need for adequate intertemporal assessment of risk and rewards propagating in time.

This chapter is organized as follows. In Section 1.1 we introduce the underlying concepts that will be used throughout the chapter. Section 1.2 provides the main contribution of our work in the context of general theory of conditional assessment indices defined on locally convex topological L^0-modules and taking values in $\bar{L}^0$. In particular, Theorem 1.12 furnishes robust representation, characterization indeed, for an upper semicontinuous conditional assessment index. This is a novel and important result, which generalizes the corresponding result obtained in the static (not conditional) setting in [30]. The road leading to Theorem 1.12, which at first sight seems to be similar to what was done in [30], has not been an easy one, as underlined by the various technical results given in the Appendix in Chapter 6 and needed to obtain this robust dual representation In particular, we provide a full duality result for conditionally increasing functions and their general left and right inverses. The main hurdle lies in the central issue of locality, that is delicate and has to be handled with outmost care. The results regarding scale invariant indices and certainty equivalents, presented in Section 1.2.3 and in Section 1.2.4, respectively, are new and useful. In Section 1.3 we apply our general theory to study DAIs for discrete time stochastic processes. This comes in two flavors. First, in Section 1.3.1, we apply the results of Section 1.2 almost verbatim, considering dynamic assessment indices mapping processes into sequences of processes, and by making a very natural choice of L^0 space to be the space of stopped processes. Analysis of the robust representation result derived in this section brings about an interesting insight regarding the nature of the locality property: indeed, requiring locality relative to $\mathscr{O}^t$ (compare Section 1.3.1) implies that α_t^t assesses only the future (relative to t) of the process and, for $s < t$, α_s^t is a function of the value at time s of the assessed process. This is a drawback as for some applications this may be an unwanted feature. To overcome this drawback we adapt in Section 1.3.2 the theory of Section 1.2 to the case of a so called path dependent DAI, which maps processes into processes. Path dependent DAIs may, in particular, help a decision maker (for instance an investor, or a regulator), who is willing to design a DAI that at each time explicitly accounts for the past evolution of the underlying process being assessed (compare Example 1.32). In Section 1.4 we study strongly time consistent path dependent assessment indices, that satisfy additional properties. The corresponding certainty equivalent is used to derive a relevant version of the dynamic programming principle, which characterizes the strong time consistency in this case. Section 1.5 provides illustrating examples that we consider both interesting and important. We examine a version of dynamic gain-to-loss ratio, which is a scale invariant DAI, and, in particular, we provide a robust representation for it. Two additional examples are given. Finally, the Appendix collects a variety of mathematical results which underlie our theory and contains proofs of auxiliary technical results stated in the main body of the chapter.

1.1 Preliminaries

We denote $L^0 = L^0(\Omega, \mathscr{G}, P)$. If not otherwise specified, the notation $[A_i] \subseteq \mathscr{G}$ stands for a countable partition $(A_i)_{i\in\mathbb{N}} \subseteq \mathscr{G}$ of Ω. The space L^0 is a lattice ordered ring on which we, throughout this chapter, consider the topology induced by the balls

$$B_\varepsilon(m) := \left\{ n \in L^0 : |m - n| \leq \varepsilon \right\}, \quad m \in L^0, \text{and } \varepsilon \in L^0_{++},$$

making L^0 to be a topological ring[1]. We refer to [43, 61], initiating the theory of L^0-modules, for further details.

From this point on, $\mathcal{X}$ denotes an L^0-module. A set $\mathcal{C} \subseteq \mathcal{X}$ is called L^0-convex if $\lambda X + (1-\lambda) Y \in \mathcal{C}$ for any $\lambda \in L^0$ with $0 \leq \lambda \leq 1$ and $X, Y \in \mathcal{C}$. By definition, $\mathcal{C}$ is σ-stable if and only if $\sum 1_{A_i} X_i \in \mathcal{C}$ for every $[A_i] \subseteq \mathscr{G}$ and $(X_i) \subseteq \mathcal{C}$. In the following, $\mathcal{K} \subseteq \mathcal{X}$ will be a σ-stable, L^0-convex cone[2] containing 0. Such an L^0-convex cone defines an L^0-module preorder[3] $\succcurlyeq$ on $\mathcal{X}$, given by $X \succcurlyeq Y$ if $X - Y \in \mathcal{K}$. We say a set $\mathcal{C} \subseteq \mathcal{X}$ is monotone with respect to $\mathcal{K}$, or just monotone if there is no ambiguity about $\mathcal{K}$, if $\mathcal{C} + \mathcal{K} = \mathcal{C}$.

[1] That is, both the addition and scalar multiplication are continuous mappings with respect to the product topology.
[2] That is, $\lambda X \in \mathcal{K}$ for any $\lambda \in L^0_{++}$ and $X \in \mathcal{K}$.
[3] That is, $\lambda X + Z \succcurlyeq \lambda Y + Z$ for any $\lambda \in L^0_{+}$ and $Z \in \mathcal{X}$, whenever $X \succcurlyeq Y$ for $X, Y \in \mathcal{X}$.

Working with (quasi)concave functions, we adopt the convention, $\infty - \infty := -\infty$ and $0 \cdot \pm\infty = 0$. We say that a function $F : \mathcal{X} \to \bar{L}^0$ is

- L^0-local if $F\left(1_A X + 1_{A^c} Y\right) = 1_A F\left(X\right) + 1_{A^c} F(Y)$;
- L^0-quasiconcave if $F(\lambda X + (1-\lambda)\, Y) \geq F(X) \wedge F(Y)$;
- L^0-concave if $F(\lambda X + (1-\lambda) Y) \geq \lambda F(X) + (1-\lambda) F(Y)$;
- monotone with respect to $\mathcal{K}$ if $F(X) \geq F(Y)$, whenever $X \succcurlyeq Y$;

for any $X, Y \in \mathcal{X}$, $\lambda \in L^0$ and $0 \leq \lambda \leq 1$, and any $A \in \mathscr{G}$. It can be shown that F is local if and only if

$$F\left(\sum 1_{A_i} X_i\right) = \sum 1_{A_i} F\left(1_{A_i} X_i\right) = \sum 1_{A_i} F\left(X_i\right), \tag{1.1}$$

for every $[A_i] \subseteq \mathscr{G}$ and $(X_i) \subseteq \mathcal{X}$, as well as if and only if

$$1_A F\left(X\right) = 1_A F\left(1_A X\right),$$

for every $A \in \mathscr{G}$ and $X \in \mathcal{X}$. A local function F of two arguments is called jointly local.

We further say that F is

- L^0-linear if F takes values in L^0 and $F\left(mX + nY\right) = mF\left(X\right) + nF\left(Y\right)$;
- positive homogeneous if $F(\lambda X) = \lambda F(X)$;
- scale invariant if $F(\lambda X) = F(X)$;
- κ-cash additive for $\kappa \in \mathcal{K} \setminus 0$ if $F(X + m\kappa) = F(X) + m$;

for any $X, Y \in \mathcal{X}$, any $m, n \in L^0$, and any $\lambda \in L^0_{++}$.

We now suppose that $\mathcal{X}$ is a locally L^0-convex topological L^0-module, compare [43, Definition 2.2]. We denote by $\mathcal{X}^*$ its L^0-dual, that is, the set of all continuous L^0-linear functionals from $\mathcal{X}$ to L^0. The L^0-dual $\mathcal{X}^*$ is an L^0-module itself. The weak topology, denoted by L^0-$\sigma\left(\mathcal{X}, \mathcal{X}^*\right)$, is the coarsest topology in $\mathcal{X}$ for which the mappings

$$X \mapsto Z\left(X\right), \quad X \in \mathcal{X},$$

are continuous for any $Z \in \mathcal{X}^*$.

For a function $F : \mathcal{X} \to \bar{L}^0$ and for $m \in \bar{L}^0$, we denote by $\mathcal{A}^m$ the corresponding upper level set, that is $\mathcal{A}^m := \{X \in \mathcal{X} : F(X) \geq m\}$. A function $F : \mathcal{X} \to \bar{L}^0$ is upper semicontinuous if its upper level sets $\mathcal{A}^m$ are closed for all $m \in \bar{L}^0$.

It was shown in [43, 61] that $F : \mathcal{X} \to \bar{L}^0$ is L^0-quasiconcave or monotone if and only if its upper level sets $\mathcal{A}^m$ are L^0-convex or monotone, for any $m \in \bar{L}^0$. It is also known that F is L^0-concave (resp. L^0-local) if and only if its hypograph $\operatorname{hypo}(F) := \{(X,m) \in \mathcal{X} \times \bar{L}^0 : \alpha(X) \geq m\}$ is L^0-convex[4] (resp. σ-stable).

A set $\mathcal{B} \subseteq L^0$ is upward directed, respectively downward directed if $X \wedge Y$, respectively $X \vee Y$, belongs to $\mathcal{B}$, for any $X, Y \in \mathcal{B}$. In case of an upward directed, respectively downward directed, set, its essential supremum, respectively essential infimum, is attained by an increasing, respectively decreasing, sequence in this set, compare [49, Appendix A5]. Similar results hold true for family of sets. If $(A_i) \subseteq \mathcal{G}$ is upward, respectively downward, directed with respect to the inclusion preorder, then there exists essential supremum[5], respectively essential infimum, $A \in \mathcal{G}$, compare [43, Lemma 2.9].

Throughout this chapter, if no confusion may arise, we will often drop the reference to L^0 for all concepts from convex analysis.

1.2 Robust Representation of Conditional Assessment Indices

In this section we follow the lines of [30], extending the setup and the results presented therein to the conditional case. In the rest of this section we fix a σ-stable cone $\mathcal{K} \subseteq \mathcal{X}$, and the monotonicity will be understood with respect to this cone.

1.2.1 Conditional Assessment Indices and Conditional Risk Acceptance Family

The main object studied in this chapter is the conditional assessment index defined as follows.

Definition 1.1. A conditional assessment index is a function $\alpha : \mathcal{X} \to \bar{L}^0$, which is local, quasiconcave, and monotone.[6]

Analogously to the one-to-one relation between risk measures and risk acceptance families discussed in [30], we also obtain a one-to-one relation, stated in Theorem 1.4, between conditional assessment indices and conditional risk acceptance families defined below.

Definition 1.2. A conditional risk acceptance family is a family $\mathcal{A} := (\mathcal{A}^m)_{m \in \bar{L}^0}$ of sets in $\mathcal{X}$, which is

- convex: $\mathcal{A}^m$ is convex, for any $m \in \bar{L}^0$;
- decreasing: $\mathcal{A}^m \subseteq \mathcal{A}^n$, for any $n, m \in \bar{L}^0$ such that $m \geq n$;

[4] Even if $\bar{L}^0$ is not an L^0-module, using the convention $\infty - \infty = \infty$ and $0 \cdot \infty = 0$ on $\bar{L}^0$ we get the analogous results.

[5] That is, if $B \in \mathcal{G}$ is such that $A_i \subseteq B \subseteq A$ for all i, it holds $P[A \Delta B] = 0$.

[6] Recall that all concepts, such as quasiconcave, local, etc., are understood in the L^0-sense.

- monotone: $\mathcal{A}^m + \mathcal{K} = \mathcal{A}^m$, for any $m \in \bar{L}^0$;
- jointly σ-stable: $\mathcal{A} = (\mathcal{A}^m)_{m \in \bar{L}^0} = \{(X, m) \in \mathcal{X} \times \bar{L}^0 : X \in \mathcal{A}^m\} \subseteq \mathcal{X} \times \bar{L}^0$ is σ-stable;
- left-continuous: for every $m \in \bar{L}^0$, the following identity holds true

$$\mathcal{A}^m = 1_{B(m)} \bigcap_{\substack{n<m \text{ on } B(m) \\ n=-\infty \text{ on } B^c(m)}} \mathcal{A}^n + 1_{B^c(m)} \mathcal{X},$$

  where $B(m) = \{m > -\infty\}$.

Remark 1.3. Note that the joint σ-stability of $\mathcal{A}$ is equivalent to the property that

$$\sum 1_{A^i} \mathcal{A}^{m_i} = \mathcal{A}^{\sum 1_{A^i} m^i},$$

for any sequence $(m^i) \subseteq \bar{L}^0$, and any $[A^i] \subseteq \mathscr{G}$. In particular, taking $m^i = m,\ i \in \mathbb{N}$, we get that $\sum 1_{A^i} \mathcal{A}^m = \mathcal{A}^m$, and consequently we obtain that the set $\mathcal{A}^m$ is σ-stable.

We are ready now to state and prove a one-to-one relationship between conditional assessment indices and conditional risk acceptance family. This result will play a central role in the proof of the robust representation theorem (compare Section 1.2.2).

Theorem 1.4. *Given a conditional assessment index α, the family $\mathcal{A}_\alpha = (\mathcal{A}_\alpha^m)_{m \in \bar{L}^0}$ of sets defined by*

$$\mathcal{A}_\alpha^m := \{X \in \mathcal{X} : \alpha(X) \geq m\}, \quad m \in \bar{L}^0, \tag{1.2}$$

is a conditional risk acceptance family.

Conversely, given a conditional risk acceptance family $\mathcal{A} = (\mathcal{A}^m)_{m \in \bar{L}^0}$, the function $\alpha_{\mathcal{A}} : \mathcal{X} \to \bar{L}^0$ defined by

$$\alpha_{\mathcal{A}}(X) := \operatorname{ess\,sup} \left\{m \in \bar{L}^0 : X \in \mathcal{A}^m\right\}, \quad X \in \mathcal{X}, \tag{1.3}$$

is a conditional assessment index.

Furthermore, with the previous notation, $\alpha_{\mathcal{A}_\alpha} = \alpha$ and $\mathcal{A}_{\alpha_{\mathcal{A}}} = \mathcal{A}$.

Remark 1.5. In the above result, $\mathcal{A}^m$ has to be indexed by $m \in \bar{L}^0$ rather than $m \in L^0$ to get a one-to-one correspondence. Indeed, suppose that our probability measure P can be extended to $\mathscr{G}_1 \supseteq \mathscr{G}$. Let $\mathcal{X} := L^p_{\mathscr{G}}(\mathscr{G}_1)$, for some $p \geq 1$, where $L^p_{\mathscr{G}}(\mathscr{G}_1)$ is defined as $L^p_{\mathcal{F}}(\mathcal{E})$ in [61, Section 4.2]. We choose an $A \in \mathscr{G}$, with $0 < P[A] < 1$. It is straightforward to check that the function

$$\alpha(X) := E[X|\mathscr{G}] 1_A - \infty 1_{A^c},$$

is a conditional assessment index. However, since $\{X \in \mathcal{X} : \alpha(X) \geq m\} = \emptyset$ for all $m \in L^0$, it follows that $\alpha_{\mathcal{A}_\alpha} = -\infty \neq \alpha$.

Similar conclusion holds true for more general and economically sound examples. Let u_1 and u_2 be utility functions, such that $1_A E[u_1(X_0)|\mathscr{G}] > -\infty$ and $1_{A^c} E[u_2(X_0)|\mathscr{G}] = -\infty$ for some $X_0 \in \mathcal{X}$, and some $A \in \mathscr{G}$ with $0 < P[A] < 1$. For example, one can choose exponential utility $u_1(x) = 1 - e^{-x}$, and log utility $u_2(x) = \ln(1 + x)$. We consider the state dependent utility function $u = 1_A u_1 + 1_{A^c} u_2$. Then, $\alpha(X) := E[u(X)|\mathscr{G}],\ X \in \mathcal{X}$, is a conditional assessment index. However, by similar arguments as above, $\alpha_{\mathcal{A}_\alpha} \neq \alpha$.

Remark 1.6. Note that a version of Theorem 1.4 has been derived in [51]. However, for a risk acceptance family we require joint σ-stability and an indexing by $\bar{L}^0$ rather than L^0; compare also Remark 1.5.

In contrast to the approach in the proof of the robust representation in [51], here the starting point for the Robust representation theorem 1.12 will be the one-to-one correspondence between conditional assessment indices and conditional risk acceptance families stated in Theorem 1.4. We also note that Theorem 1.4 is the conditional version of [30, Theorem 1.7].

Proof. Step 1: Let α be a conditional assessment index, and let the family of acceptance sets $\mathcal{A}_\alpha^m$, where $m \in \bar{L}^0$, be defined as in (1.2). By definition, $\mathcal{A}_\alpha$ is decreasing. Furthermore, for any $m \in \bar{L}^0$, the set $\mathcal{A}_\alpha^m$ is convex and monotone, since it is an upper level set of a quasiconcave and monotone function.

Next we show that $\mathcal{A}_\alpha$ is jointly σ-stable. Let $[A_i] \subseteq \mathscr{G}$ and $(X_i, m_i)_{i\in\mathbb{N}} \subseteq \mathcal{A}_\alpha$, in particular $\alpha(X_i) \geq m_i,\ i \in \mathbb{N}$. By definition of $\mathcal{A}_\alpha$, by locality of α, and by (1.1), it follows that $(X, m) := \sum 1_{A_i}(X_i, m_i) = (\sum 1_{A_i} X_i, \sum 1_{A_i} m_i)$ fulfills

$$\alpha(X) = \alpha\left(\sum 1_{A_i} X_i\right) = \sum 1_{A_i} \alpha\left(1_{A_i} X_i\right) = \sum 1_{A_i} \alpha\left(X_i\right) \geq \sum 1_{A_i} m_i = m,$$

so that $(X, m) \in \mathcal{A}_\alpha$, which shows the joint σ-stability of $\mathcal{A}_\alpha$.

Finally we prove the left-continuity of $\mathcal{A}_\alpha$. Let $m \in \bar{L}^0$ and $B(m) = \{m > -\infty\}$. We start by observing that

$$\bigcap_{\substack{n<m \text{ on } B(m)\\ n=-\infty \text{ on } B^c(m)}} \mathcal{A}^n = \bigcap_{\substack{n<m\\ \text{on } B(m)}} \mathcal{A}_\alpha^{1_{B(m)}n - 1_{B^c(m)}\infty}.$$

Next, we note that

$$
\begin{aligned}
& 1_{B(m)} \bigcap_{\substack{n<m \\ \text{on } B(m)}} \mathcal{A}_\alpha^{1_A n - 1_{B^c(m)}\infty} + 1_{B^c(m)}\mathcal{X} \\
=& 1_{B(m)} \bigcap_{\substack{n<m \\ \text{on } B(m)}} \left\{X \in \mathcal{X} : \alpha(X) \geq 1_{B(m)} n - 1_{B^c(m)}\infty\right\} \\
+& 1_{B^c(m)} \left\{Y \in \mathcal{X} : \alpha(Y) \geq -\infty\right\} \\
=& \big\{1_{B(m)}X + 1_{B^c(m)}Y \in \mathcal{X} : \alpha(X) \geq 1_{B(m)} n - 1_{B^c(m)}\infty \text{ for all } n < m \text{ on } B(m), \\
& \qquad \text{and } \alpha(Y) \geq -\infty\big\}.
\end{aligned}
$$

Since α is local, it follows that

$$
\begin{aligned}
& \big\{1_{B(m)}X + 1_{B^c(m)}Y \in \mathcal{X} : \alpha(X) \geq 1_{B(m)} n - 1_{B^c(m)}\infty \\
& \qquad \text{for all } n < m \text{ on } B(m) \text{ and } \alpha(Y) \geq -\infty\big\} \\
&= \big\{1_{B(m)}X + 1_{B^c(m)}Y \in \mathcal{X} : 1_{B(m)}\alpha(X) + 1_{B^c(m)}\alpha(Y) = \alpha(1_{B(m)}X + 1_{B^c(m)}Y) \\
& \qquad \geq 1_{B(m)} n - 1_{B^c(m)}\infty \text{ for all } n < m \text{ on } B(m)\big\} \\
&= \big\{X \in \mathcal{X} : \alpha(X) \geq 1_{B(m)} n - 1_{B^c(m)}\infty \text{ for all } n < m \text{ on } B(m)\big\} = \mathcal{A}^m.
\end{aligned}
$$

Hence, left-continuity of $\mathcal{A}_\alpha$ is proved.

Step 2: Conversely, assume $\mathcal{A} = (\mathcal{A}^m)_{m \in \bar{L}^0}$ to be an acceptance family, and consider $\alpha_{\mathcal{A}}$ defined as in (1.3). First we prove that $\alpha_{\mathcal{A}}$ is monotone. Consider $X, Y \in \mathcal{X}$ such that $X \succcurlyeq Y$. By monotonicity of $\mathcal{A}$, if $Y \in \mathcal{A}^m$, then $X \in \mathcal{A}^m$. Hence

$$
\left\{m \in \bar{L}^0 \,:\, Y \in \mathcal{A}^m\right\} \subseteq \left\{m \in \bar{L}^0 \,:\, X \in \mathcal{A}^m\right\}.
$$

Taking $\operatorname{ess\,sup}$ of both sides in the last inclusion, the monotonicity of $\alpha_{\mathcal{A}}$ follows.

Next we will show that $\alpha_{\mathcal{A}}$ is quasiconcave. In order to do this, we consider $X, Y \in \mathcal{X}$ and we let $m, n \in \bar{L}^0$ be such that $X \in \mathcal{A}^m$ and $Y \in \mathcal{A}^n$. Such m, n exist, since by the left-continuity of $\mathcal{A}$ we have $\mathcal{A}^{-\infty} = \mathcal{X}$. Next, we set $\tilde{m} = m \wedge n$, and from the decreasing property of $\mathcal{A}$ we conclude that $X, Y \in \mathcal{A}^{\tilde{m}}$. Now, we choose $\lambda \in L^0$ such that $0 \leq \lambda \leq 1$. By convexity of $\mathcal{A}$, we get that the convex combination $Z := \lambda X + (1-\lambda)Y \in \mathcal{A}^{\tilde{m}}$, and hence $\alpha_{\mathcal{A}}(Z) \geq \tilde{m}$. Consequently,

$$
\alpha_{\mathcal{A}}(Z) \geq \operatorname{ess\,sup}\{m \in \bar{L}^0 \,:\, X \in \mathcal{A}^m\} \wedge \operatorname{ess\,sup}\{n \in \bar{L}^0 \,:\, Y \in \mathcal{A}^n\}.
$$

Thus, we conclude that $\alpha_{\mathcal{A}}(Z) \geq \alpha_{\mathcal{A}}(X) \wedge \alpha_{\mathcal{A}}(Y)$, which proves quasiconcavity of $\alpha_{\mathcal{A}}$.

It remains to prove locality of $\alpha_{\mathcal{A}}$. For this, let $A \in \mathscr{G}$, $X \in \mathcal{X}$, and consider $m \in \bar{L}^0$ such that $X \in \mathcal{A}^m$. Again, such m exists since by the left-continuity of $\mathcal{A}$ we have $\mathcal{A}^{-\infty} = \mathcal{X}$.

From σ-stability of $\mathcal{A}$, and from the fact that $0 \in \mathcal{A}^{-\infty}$, we have $1_A X \in \mathcal{A}^{1_A m - 1_{A^c}\infty}$, which implies that $\alpha_{\mathcal{A}}(1_A X) \geq 1_A m - 1_{A^c}\infty$. Hence, $1_A \alpha_{\mathcal{A}}(1_A X) \geq 1_A m$, and thus, taking the essential supremum with respect to m in this inequality, we get

$$1_A \alpha_{\mathcal{A}}(1_A X) \geq 1_A \alpha_{\mathcal{A}}(X). \tag{1.4}$$

Now, let $n \in \bar{L}^0$ such that $1_A X \in \mathcal{A}^n$. Since $X \in \mathcal{A}^{-\infty}$, we get by σ-stability of $\mathcal{A}$, compare Remark 1.3,

$$X = 1_A(1_A X) + 1_{A^c} X \in \mathcal{A}^{1_A n - 1_{A^c}\infty}.$$

This implies that $\alpha_{\mathcal{A}}(X) \geq 1_A n - 1_{A^c}\infty$, and consequently $1_A \alpha_{\mathcal{A}}(X) \geq 1_A n$. Taking the essential supremum with respect to n in the last inequality, we get $1_A \alpha_{\mathcal{A}}(X) \geq 1_A \alpha(1_A X)$, which, jointly with (1.4), demonstrates locality of $\alpha_{\mathcal{A}}$.

Thus $\alpha_{\mathcal{A}}$ is a conditional assessment index.

Step 3: We finally prove the last statement of Theorem 1.4. Assume that α is a conditional assessment index. Then, $\mathcal{A}_\alpha$ is a conditional risk acceptance family, and therefore $\alpha_{\mathcal{A}_\alpha}$ is a conditional assessment index. Note that for any $X \in \mathcal{X}$ we have

$$\alpha_{\mathcal{A}_\alpha}(X) = \operatorname{ess\,sup}\left\{m \in \bar{L}^0 \,:\, X \in \mathcal{A}_\alpha^m\right\} = \operatorname{ess\,sup}\left\{m \in \bar{L}^0 \,:\, \alpha(X) \geq m\right\} = \alpha(X),$$

and so $\alpha = \alpha_{\mathcal{A}_\alpha}$.

Assume now that $\mathcal{A}$ is a conditional risk acceptance family. We will show that $\mathcal{A}^m_{\alpha_{\mathcal{A}}} = \mathcal{A}^m$ for any $m \in \bar{L}^0$, from which we deduce that $\mathcal{A}_{\alpha_{\mathcal{A}}} = \mathcal{A}$.

If $m = -\infty$, then $\mathcal{A}^{-\infty}_{\alpha_{\mathcal{A}}} = \{X \in \mathcal{X} : \alpha_{\mathcal{A}}(X) \geq -\infty\} = \mathcal{X}$, and by the left-continuity of $\mathcal{A}$ we get that $\mathcal{A}^{-\infty}_{\alpha_{\mathcal{A}}} = \mathcal{A}^{-\infty}$.

Next, assume that $m > -\infty$. Given $\varepsilon \in L^0_{++}$, we claim that $\alpha_{\mathcal{A}}(X) \geq m$ implies that $X \in \mathcal{A}^{m-\varepsilon}$. Indeed, $\mathcal{A}$ being jointly σ-stable, $\{n \in \bar{L}^0 : X \in \mathcal{A}^n\}$ is upward directed. Hence, there exists an increasing sequence $(n_i) \subseteq \{n \in \bar{L}^0 : X \in \mathcal{A}^n\}$, such that $n_i \uparrow \alpha_{\mathcal{A}}(X)$. Let $A_i := \{n_i \geq m - \varepsilon\}$ and $B_i := A_i \setminus A_{i-1}$, for $i \in \mathbb{N}$, and put $B_0 := A_0$. Then $[B_i] \subseteq \mathscr{G}$, and $X \in \mathcal{A}^{n_i}$ for every $i \in \mathbb{N}$. By σ-stability of $\mathcal{A}$, we get that $\sum 1_{B_i}(X, n_i) = (X, \sum 1_{B_i} n_i) \in \mathcal{A}$. However, by construction, $\tilde{m} := \sum 1_{B_i} n_i \geq m - \varepsilon$, and thus $X \in \mathcal{A}^{m-\varepsilon}$. Consequently, we deduce

$$\begin{aligned}
\mathcal{A}^m_{\alpha_{\mathcal{A}}} &= \{X \in \mathcal{X} : \alpha_{\mathcal{A}}(X) \geq m\} \\
&= \left\{X \in \mathcal{X} : \operatorname{ess\,sup}\{n \in \bar{L}^0 : X \in \mathcal{A}^n\} \geq m\right\} \\
&= \left\{X \in \mathcal{X} : X \in \mathcal{A}^{m-\varepsilon} \text{ for all } \varepsilon \in L^0_{++}\right\} \\
&= \bigcap_{n<m} \mathcal{A}^n.
\end{aligned}$$

Finally, since in view of the left-continuity of $\mathcal{A}$ we have $\cap_{n<m}\mathcal{A}^n = \mathcal{A}^m$, compare Remark 1.3, we obtain that $\mathcal{A}^m_{\alpha_{\mathcal{A}}} = \mathcal{A}^m$.

For the general case $m \in \bar{L}^0$, we write $m = -1_{A^c}\infty + 1_A m$, with $A = \{m > -\infty\}$, and using σ-stability we conclude that $\mathcal{A}^m_{\alpha_{\mathcal{A}}} = \mathcal{A}^m$. Thus, $\mathcal{A}_{\alpha_{\mathcal{A}}} = \mathcal{A}$.

□

1.2.2 Robust representation

In this subsection we prove a robust representation theorem for conditional assessment indices, which is one of the main results of this chapter. From now on we suppose that $\mathcal{X}$ is a conditional locally convex topological module.[7] We further suppose that the σ-stable cone $\mathcal{K}$ is closed, and we define the associated polar cone by

$$\mathcal{K}^\circ := \{X^* \in \mathcal{X}^* : \langle X^*, X\rangle \geq 0 \text{ for all } X \in \mathcal{K}\}.$$

We will now introduce two concepts, which are pivotal for our studies.

Definition 1.7. A conditional risk function is a function $R : \mathcal{K}^\circ \times \bar{L}^0 \to \bar{L}^0$ such that

(i) it is jointly local;

(ii) the map $s \mapsto R(X^*, s)$ is increasing and right-continuous for any $X^* \in \mathcal{K}^\circ$.

A conditional risk function R is called minimal if

(iii) it is jointly quasiconvex, and $R(\lambda X^*, s) = R(X^*, s/\lambda)$ for all $\lambda \in L^0_{++}$;

(iv) it has a uniform asymptotic maximum, which means

$$\operatorname*{ess\,sup}_{s\in L^0} R(X^*, s) = \operatorname*{ess\,sup}_{s\in L^0} R(Y^*, s),$$

for any $X^*, Y^* \in \mathcal{K}^\circ$;

(v) the left-continuous version (in the second argument) $R^-(X^*, s)$ is jointly lower semicontinuous.

The set of all conditional minimal risk functions is denoted by $\mathcal{R}^{\min}$.

Remark 1.8. Note that Condition (iv) is equivalent to

(iv′) it has a uniform asymptotic maximum, which means

$$R^-(X^*, \infty) = R^-(Y^*, \infty),$$

for any $X^*, Y^* \in \mathcal{K}^\circ$.

[7] Compare [43] and Appendix 1.6 for background on conditional modules.

Definition 1.9. A conditional maximal penalty function is a function $\pi : \mathcal{K}^\circ \times \bar{L}^0 \to \bar{L}^0$ such that

(a) it is jointly local;

(b) the map $m \mapsto \pi(X^*, m)$ is increasing and right-continuous for any $X^* \in \mathcal{K}^\circ$;

(c) it is positive homogeneous in the first argument[8] and concave in the first argument;

(d) it is maximal invariant, that is, if $\pi(X^*, m) = \infty$ for some $X^* \in \mathcal{K}^\circ$ and $m \in \bar{L}^0$, then $\pi(Y^*, m) = \infty$ for all $Y^* \in \mathcal{K}^\circ$;

(e) it is upper semicontinuous in the first argument.

The set of all conditional maximal penalty functions is denoted by $\mathcal{P}^{\max}$.

Proposition 1.10. *The set of conditional minimal risk functions $R \in \mathcal{R}^{\min}$ and the set of conditional maximal penalty functions $\pi \in \mathcal{P}^{\max}$ are related in the following manner*

$$\pi^{(-1,r)}(X^*, s) \in \mathcal{R}^{\min},$$
$$R^{(-1,r)}(X^*, m) \in \mathcal{P}^{\max},$$

where $\pi^{(-1,r)}(X^*, s)$ and $R^{(-1,r)}(X^*, m)$ denote the right-inverse[9] in the second argument for fixed $X^* \in \mathcal{X}^*$. Moreover, the relationship is one-to-one.

The proof of this proposition is deferred to the Appendix 1.6.3.

Proposition 1.11. *Let $\mathcal{C} \subseteq \mathcal{X}$ be a closed, convex, monotone and σ-stable set. Then, there exists a unique local function $\pi : \mathcal{K}^\circ \to \bar{L}^0$ such that it is*

(a) positive homogeneous and concave;

(b) maximal invariant;

(c) upper semicontinuous,

and such that

$$X \in \mathcal{C} \iff \langle X, X^* \rangle \geq \pi(X^*), \quad \text{for all } X^* \in \mathcal{K}^\circ. \tag{1.5}$$

Moreover, this function is explicitly given by the relation

$$\pi(X^*) = \chi^\star_{\mathcal{C}}(X^*) := \operatorname*{ess\,inf}_{X \in \mathcal{C}} \langle X^*, X \rangle, \quad \text{for all } X^* \in \mathcal{K}^\circ.$$

[8] $\pi(\lambda X^*, m) = \lambda \pi(X^*, m)$ for all $\lambda \in L^0_{++}$ and $X^*, m \in \mathcal{K}^\circ, \bar{L}^0$.

[9] For further details apply Definition 1.45 for the second argument of π and R, respectively.

The proof of this proposition is also deferred to the Appendix 1.6.4.

Finally, we are in the position to prove the main result of this section.

Theorem 1.12.

(i) Let $\alpha : \mathcal{X} \to \bar{L}^0$ be an upper semicontinuous conditional assessment index. Then, α has the robust representation of the form

$$\alpha(X) = \operatorname*{ess\,inf}_{X^* \in \mathcal{K}^\circ} R\left(X^*, \langle X^*, X\rangle\right), \tag{1.6}$$

for a unique $R \in \mathcal{R}^{\min}$;

(ii) For any conditional risk function R, the right hand-side of (1.6) defines an upper semicontinuous conditional assessment index.

Proof. *(i)* According to Theorem 1.4,

$$\alpha(X) = \operatorname{ess\,sup}\left\{m \in \bar{L}^0 \,:\, X \in \mathcal{A}^m\right\}, \quad X \in \mathcal{X}, \tag{1.7}$$

where $\mathcal{A} = (\mathcal{A}^m)_{m \in \bar{L}^0}$ is the corresponding conditional risk acceptance family in the sense of (1.2). In particular, each of the sets $\mathcal{A}^m$, $m \in \bar{L}^0$, is monotone, convex, and, in view of Remark 1.3, it is also σ-stable. In addition, since α is upper semicontinuous then each set $\mathcal{A}^m$, $m \in \bar{L}^0$ is closed. Thus, defining $\pi : \mathcal{K}^\circ \times \bar{L}^0 \to \bar{L}^0$ by

$$\pi(X^*, m) := \operatorname*{ess\,inf}_{X \in \mathcal{A}^m} \langle X^*, X\rangle,$$

we have by Proposition 1.11 that π satisfies properties (c)-(e) of Definition 1.9. Moreover, from (1.5), we conclude that

$$X \in \mathcal{A}^m \quad \Longleftrightarrow \quad \langle X, X^*\rangle \geq \pi(X^*, m) \text{ for all } X^* \in \mathcal{K}^\circ,$$

which in combination with (1.7) yields

$$\alpha(X) = \operatorname{ess\,sup}\left\{m \in \bar{L}^0 \,:\, \langle X, X^*\rangle \geq \pi(X^*, m), \text{ for all } X^* \in \mathcal{K}^\circ\right\}.$$

Furthermore, π fulfills (a) of Definition 1.9. Indeed, let $X^*, Y^* \in \mathcal{K}^\circ$, $m, \tilde{m} \in \bar{L}^0$ and $A \in \mathscr{G}$. Since, $\mathcal{A}$ is jointly σ-stable, it follows that $\mathcal{A}^{1_A m + 1_{A^c}\tilde{m}} = 1_A \mathcal{A}^m + 1_{A^c}\mathcal{A}^{\tilde{m}}$.

Hence

$$\begin{aligned}\pi\left(1_A X^* + 1_{A^c} Y^*, 1_A m + 1_{A^c}\tilde{m}\right) &= \operatorname*{ess\,inf}_{\tilde{X}\in\mathcal{A}^{1_A m + 1_{A^c}\tilde{m}}} \langle 1_A X^* + 1_{A^c} Y^*, \widetilde{X}\rangle \\ = \operatorname*{ess\,inf}_{\widetilde{X}\in 1_A\mathcal{A}^m + 1_{A^c}\mathcal{A}^{\tilde{m}}} \langle 1_A X^* + 1_{A^c} Y^*, \widetilde{X}\rangle &= \operatorname*{ess\,inf}_{X\in\mathcal{A}^m, Y\in\mathcal{A}^{\tilde{m}}} \langle 1_A X^* + 1_{A^c} Y^*, 1_A X + 1_{A^c} Y\rangle \\ = \operatorname*{ess\,inf}_{X\in\mathcal{A}^m, Y\in\mathcal{A}^{\tilde{m}}} \left(1_A\langle X^*, X\rangle + 1_{A^c}\langle Y^*, Y\rangle\right) &= 1_A \operatorname*{ess\,inf}_{X\in\mathcal{A}^m}\langle X^*, X\rangle + 1_{A^c}\operatorname*{ess\,inf}_{Y\in\mathcal{A}^{\tilde{m}}}\langle Y^*, Y\rangle \\ &= 1_A\pi\left(X^*, m\right) + 1_{A^c}\pi\left(Y^*, \tilde{m}\right),\end{aligned}$$

hence π is jointly local.

Since the map $m \mapsto \pi(\cdot, m)$ is increasing[10], the left- and right-continuous version of it, say, π^- and π^+ respectively, are given as in (1.36) and (1.37). Moreover, it is rather clear that π^+ fulfills[11] the conditions (a)-(e) of Definition 1.9, and thus $\pi^+ \in \mathcal{P}^{\max}$.

Next we show that

$$\alpha(X) = \beta^-(X) = \beta^+(X), \quad X \in \mathcal{X}, \tag{1.8}$$

where

$$\beta^-(X) := \operatorname{ess\,sup}\left\{m \in \bar{L}^0 \;:\; \langle X, X^*\rangle \geq \pi^-(X^*, m) \text{ for all } X^* \in \mathcal{K}^\circ\right\}, \tag{1.9}$$

$$\beta^+(X) := \operatorname{ess\,sup}\left\{m \in \bar{L}^0 \;:\; \langle X, X^*\rangle \geq \pi^+(X^*, m) \text{ for all } X^* \in \mathcal{K}^\circ\right\}. \tag{1.10}$$

Since $\pi^-(X^*, m) \leq \pi(X^*, m) \leq \pi^+(X^*, m)$ for all $X^*, m \in \mathcal{K}^\circ \times \bar{L}^0$, it follows that

$$\beta^-(X) \geq \alpha(X) \geq \beta^+(X), \quad X \in \mathcal{X}. \tag{1.11}$$

If $\beta^-(X)$ is equal to $-\infty$ on some set $A \in \mathscr{G}_+$, then equality (1.8) holds true on A. Hence, using locality, it is enough to prove that (1.8) holds true for $\beta^-(X) > -\infty$. By the definition of β^-, there exists an increasing sequence $(m^n) \subseteq L^0$ converging to $\beta^-(X)$, and such that $m^n < m^{n+1} < \beta^-(X)$. By the definition of the left-and right-continuous version of an increasing function, we get $\pi^+(X^*, m^n) \leq \pi^-\left(X^*, m^{n+1}\right)$, for all $X^* \in \mathcal{K}^\circ$, and all $n \in \mathbb{N}$. Hence, $m^n \leq \beta^+(X)$ for all $n \in \mathbb{N}$, and therefore $\beta^+(X) \geq \beta^-(X)$. This, combined with (1.11), implies (1.8).

Denote by R the right-inverse of π^+. By Proposition 1.49, compare Remark 1.50, we have that $R = (\pi^-)^{(-1,r)}$. Thus, by (1.8) and (1.45) we conclude that

$$\alpha(X) = \operatorname{ess\,sup}\left\{m \in \bar{L}^0 : R\left(X^*, \langle X, X^*\rangle\right) \geq m \text{ for all } X^* \in \mathcal{K}^\circ\right\},$$

[10] Due to the fact that $\mathcal{A}$ is decreasing.

[11] In particular, notice that an essential infimum of a family of upper semicontinuous functions is an upper semicontinuous, and in view of (1.37) π^+ is upper semicontinuous.

and, consequently,

$$\alpha(X) = \operatorname*{ess\,sup}\left\{m \in \bar{L}^0 : \operatorname*{ess\,inf}_{X^* \in \mathcal{K}^\circ} R\left(X^*, \langle X, X^*\rangle\right) \geq m\right\} = \operatorname*{ess\,inf}_{X^* \in \mathcal{K}^\circ} R\left(X^*, \langle X^*, X\rangle\right).$$

Finally, we show the uniqueness of $R \in \mathcal{R}^{\min}$. Using Proposition 1.10 and (1.8), it is sufficient to show that

$$\alpha(X) = \operatorname*{ess\,sup}\left\{m \in \bar{L}^0 \,:\, \langle X, X^*\rangle \geq \tilde{\pi}\left(X^*, m\right) \text{ for all } X^* \in \mathcal{K}^\circ\right\}. \tag{1.12}$$

holds true for a unique $\tilde{\pi} \in \mathcal{P}^{\max}$. We assume, that (1.12) is satisfied for $\pi^i \in \mathcal{P}^{\max},\ i = 1, 2$. For every $n \in \bar{L}^0$ and $i = 1, 2$, we consider the sets

$$\begin{aligned} \mathcal{A}^{n,i} &:= \left\{X \in \mathcal{X} \,:\, \langle X^*, X\rangle \geq \pi^i\left(X^*, n\right) \text{ for all } X^* \in \mathcal{K}^\circ\right\} \\ &= \bigcap_{X^* \in \mathcal{K}^\circ} \{X \in \mathcal{X} : \langle X^*, X\rangle \geq \pi^i(X^*, n)\}. \end{aligned} \tag{1.13}$$

For every $X^* \in \mathcal{K}^\circ,\ m \in \bar{L}^0$, the set $\{X \in \mathcal{X} : \langle X^*, X\rangle \geq m\}$ is clearly closed, convex, and σ-stable and monotone. By (1.13), we conclude that $\mathcal{A}^{n,i}$ are closed, convex, monotone and σ-stable, for every $n \in \bar{L}^0$ and $i = 1, 2$. Let $A = \{m = \infty\}$. By Proposition 1.11, we have that, for $i = 1, 2$,

$$\pi^i\left(X^*, m\right) = \operatorname*{ess\,inf}_{\substack{n \geq m \\ n > m \text{ on } A}} \pi^i\left(X^*, n\right) = \operatorname*{ess\,inf}_{\substack{n \geq m \\ n > m \text{ on } A}} \operatorname*{ess\,inf}_{X \in \mathcal{A}^{n,i}} \langle X^*, X\rangle = \operatorname*{ess\,inf}_{X \in \bigcup\limits_{\substack{n \geq m \\ n > m \text{ on } A}} \mathcal{A}^{n,i}} \langle X^*, X\rangle. \tag{1.14}$$

on A^c.

Next we will show that

$$\bigcup_{\substack{n \geq m \\ n > m \text{ on } A}} \mathcal{A}^{n,i} = \left\{X \in \mathcal{X} : \alpha(X) \geq m \text{ and } \alpha\left(X\right) > m \text{ on } A\right\}, \quad i = 1, 2. \tag{1.15}$$

If X belongs the left hand side, then $X \in A^{n_0,i}$ for some $n_0 \geq m$ with $n_0 > m$ on A, and hence, by (1.8) together with (1.10), we get that $\alpha(X) \geq n_0$, and consequently we conclude that X belongs to the right hand side. Conversely, if $\alpha\left(X\right) \geq m$ with $\alpha(X) > m$ on A, then by (1.8) together with (1.10), there exists $n_0 \geq m$ with $n_0 > m$ on A such that $\langle X^*, X\rangle \geq \pi^i\left(X^*, n_0\right)$ for all $X^* \in \mathcal{K}^\circ$. Hence, $X \in A^{n_0,i}$ and therefore X is in the left hand side of (1.15). Finally, (1.14) combined with (1.15) imply that $\pi^1 = \pi^2$ on A^c. Since π^i are right-continuous, $\pi^1 = \pi^2 = \infty$ on A, and thus $\pi^1 = \pi^2$.

(ii) If the function R is a conditional risk function, meaning it satisfies (i) and (ii) from Definition 1.7, it follows immediately that for every $X^* \in \mathcal{K}^\circ$, the function

$R(X^*, \langle X^*, \cdot \rangle)$ is local, quasiconcave, monotone, and upper-semicontinuous. All these properties are preserved under ess inf, and this concludes the proof.

□

Remark 1.13. Similarly to [30], if there exists $\kappa \in \mathcal{K}$ such that $\langle X^*, \kappa \rangle > 0$ for any $X^* \in \mathcal{K}^\circ$, the robust representation (1.6) can be achieved on the normalized set

$$\mathcal{K}^\circ_\kappa := \{X^* \in \mathcal{K}^\circ : \langle X^*, \pi \rangle = 1\},$$

for a unique minimal risk function $R : \mathcal{K}^\circ_\kappa \times L^0 \to \bar{L}^0$. In this case the condition (iii) from Definition 1.7 is replaced by

$(iii)'$ it is jointly quasiconvex.

Additional properties of α are shared by the corresponding dual minimal risk function, as stated in the following result.

Proposition 1.14. *An upper semicontinuous assessment index α is concave, positive homogeneous, scale invariant, or κ-cash additive if and only if the corresponding minimal risk function R is convex, positive homogeneous, scale invariant or κ-cash additive in the second argument, respectively.*

The proof is similar to that from the static case (compare [29, 30]), and we omit it here.

1.2.3 Scale Invariant Conditional Assessment Indices

In this section we specify how the robust representation looks like in the specific case of scaling invariance. Note that the acceptance sets $\mathcal{A}^m$, $m \in L^0$, corresponding to a scale invariant assessment index are closed and convex cones. We denote their polar sets as

$$\mathcal{A}^{m,\circ} := \{X^* \in \mathcal{X}^* : \langle X^*, X \rangle \geq 0 \text{ for all } X \in \mathcal{A}^m\}, \quad m \in \bar{L}^0.$$

Proposition 1.15. *Let $\alpha : \mathcal{X} \to \bar{L}^0$ be an upper semicontinuous scale invariant conditional assessment index. Then, the unique conditional minimal risk function $R \in \mathcal{R}_{\max}$ from the representation* (1.6) *has the form*

$$R(X^*, s) = \begin{cases} -\infty & \text{on } \{s = -\infty\} \\ \operatorname{ess\,inf}\left\{m \in \bar{L}^0 : X^* \in \mathcal{A}^{m,\circ}\right\} & \text{on } \{-\infty < s < 0\} \\ +\infty & \text{on } \{s \geq 0\} \end{cases}, \quad X^* \in \mathcal{K}^\circ, s \in \bar{L}^0. \tag{1.16}$$

Proof. Similar to Theorem 1.12.(i), we consider the function

$$\pi\left(X^{*}, m\right) := \operatorname*{ess\,inf}_{X \in \mathcal{A}^{m}} \langle X^{*}, X\rangle = \chi^{\star}_{\mathcal{A}^{m}}\left(X^{*}\right), \tag{1.17}$$

where the last equality follows as in (1.54). Since $\mathcal{A}^m$ is a cone, it follows that

$$\chi^{\star}_{\mathcal{A}^{m}} = \chi_{\mathcal{A}^{m,\circ}}, \tag{1.18}$$

for any $m \in \bar{L}^0$.[12] Indeed, by definition, $X^* \in \mathcal{A}^{m,\circ}$ if and only if $\langle X^*, X\rangle \geq 0$ for every $X \in \mathcal{A}^m$. Using the fact that $\mathcal{A}^m$ is a cone, we scale X with $\lambda \in L^0_{++}$ converging to 0 in the essential infimum (1.17) It follows that $X^* \in \mathcal{A}^{m,\circ}$ if and only if $\chi^{\star}_{\mathcal{A}^m}(X^*) = 0$. Reversely, if $1_B X^* \notin 1_B \mathcal{A}^{m,\circ}$ for every $B \in \mathscr{G}_+$, it follows by definition of $\mathcal{A}^{m,\circ}$ that there exists $X \in \mathcal{A}^{m,\circ}$ such that $\langle X^*, X\rangle < 0$. Scaling with $\lambda \in L^0_{++}$ tending to ∞, it follows that $1_B X^* \notin 1_B \mathcal{A}^{m,\circ}$ for every $B \in \mathscr{G}_+$ if and only if $\pi(X^*, m) = -\infty$. By locality, and definition of $\chi_{\mathcal{A}^{m,\circ}}$, we therefore deduce that equation (1.18) holds.

Finally, we need to show that R given by (1.16) is the conditional right-inverse of π in the second argument. It holds that $\chi_{\mathcal{A}^{m,\circ}}$ takes only 0 and ∞ as values. For $X^* = 0$, it clearly holds $R(0, s) = \infty$ on $\{s \geq 0\}$ and $-\infty$ on $\{s < 0\}$ which corresponds to Relation (1.16). Reversely, if $1_A X^* \neq 0$ for every $A \in \mathcal{G}_+$, it follows that $\operatorname{ess\,inf}_{m \in \bar{L}^0} \chi_{\mathcal{A}^{m,\circ}}(X^*) = \chi_{\mathcal{X}^\circ}(X^*) = \chi_{\{0\}}(X^*) = -\infty$. Hence applying the definition of the right inverse, it follows that

$$\begin{aligned}
R\left(X^{*}, s\right) &= -\infty 1_{\{s=-\infty\}} + 1_{\{s>-\infty\}} \operatorname{ess\,inf}\left\{m \in \bar{L}^{0} : \chi_{\mathcal{A}^{m,\circ}}(X^{*}) > s \text{ on } \{s > -\infty\}\right\}\\
&= -1_{\{s=-\infty\}}\infty + 1_{\{s\geq 0\}}\infty + 1_{\{-\infty<s<0\}} \operatorname{ess\,inf}\left\{m \in \bar{L}^{0} : \chi_{\mathcal{A}^{m,\circ}}(X^{*}) > s\right\}\\
&= -1_{\{s=-\infty\}}\infty + 1_{\{s\geq 0\}}\infty + 1_{\{-\infty<s<0\}} \operatorname{ess\,inf}\left\{m \in \bar{L}^{0} : X^{*} \in \mathcal{A}^{m,\circ}\right\}.
\end{aligned}$$

Using stability for the general $X^* \in \mathcal{K}^\circ$ yields the representation 1.16. □

1.2.4 Certainty Equivalent

In Cheridito and Kupper [20], a concept of certainty equivalent was studied in the context of risk measures. Here, we carry out an analogous study with regard to conditional assessment indices. In Section 1.4, we will make a crucial use of the concept of certainty equivalent in studying the (strong) time consistency of assessment indices for processes. Throughout this section we fix $\kappa \in \mathcal{K} \setminus \{0\}$.

Definition 1.16. A κ-conditional certainty equivalent of a conditional assessment index α is a local functional $C : \mathcal{X} \to L^0$ such that

$$\alpha(C(X)\kappa) = \alpha(X), \quad X \in \mathcal{X}. \tag{1.19}$$

[12] Note that this states the conditional version of the Bipolar theorem.

A natural candidate for the conditional certainty equivalent of a conditional assessment index α is given by

$$C(X) := \operatorname{ess\,inf}\left\{m \in L^0 : \alpha(m\kappa) \geq \alpha(X)\right\}, \quad X \in \mathcal{X}. \tag{1.20}$$

Remark 1.17. However, in general, definition (1.20), even though natural, may not produce a valid certainty equivalent. In particular, if α is a scale invariant assessment index, then $C(X)$ defined as in (1.20), will take values only 0 and $-\infty$, and (1.19) will not be satisfied, in general. Indeed, for simplicity assume that $\mathcal{K} = L^0_+$ and $\kappa = 1$, and let C be defined as in (1.20). For sufficiently large $m > 0$, we have that $m \geq X$, and by monotonicity of α, we deduce that $\alpha(m) \geq \alpha(X)$. Hence, using scale invariance of α, we conclude that $C(X) \leq 0$, and consequently

$$C(X) = \operatorname{ess\,inf}\left\{m \in L^0 : m \leq 0, \text{ and } \alpha(m) \geq \alpha(X)\right\}, \quad X \in \mathcal{X}.$$

Using scale invariance of α , we conclude that $C(X)$ takes only the values 0 and $-\infty$.

With (1.20) in mind, we thus need to find sufficient conditions on index α ensuring that (1.20) indeed defines a certainty equivalent.

Definition 1.18. A conditional assessment index α is

- κ-bounded, if for any $X \in \mathcal{X}$, there exist $m_1, m_2 \in L^0$ satisfying
$$\alpha(m_1\kappa) < \alpha(X) \leq \alpha(m_2\kappa). \tag{1.21}$$
- κ-strictly increasing, if $\alpha(m\kappa) > \alpha(m'\kappa)$ on $A \in \mathscr{G}$, whenever $m, m' \in L^0$ and $m > m'$ on A.
- κ-sensitive, if for $m \in L^0$ and $Y \in \mathcal{X}$ with $\alpha(m\kappa) > \alpha(Y)$ on some $A \in \mathscr{G}$, there exists an $\varepsilon \in L^0_+$ with $\varepsilon > 0$ on A, such that
$$\alpha((m-\varepsilon)\kappa) \geq \alpha(Y), \quad \text{ on } A.$$

Proposition 1.19. *Let $\alpha : \mathcal{X} \to \bar{L}^0$ be a κ-sensitive and κ-bounded upper semicontinuous conditional assessment index. Then, C defined as in* (1.20) *is a κ-conditional certainty equivalent and*

$$\alpha\left(X\right) \geq \alpha\left(Y\right) \quad \Longleftrightarrow \quad C\left(X\right) \geq C\left(Y\right), \quad X, Y \in \mathcal{X}. \tag{1.22}$$

In this case, C is itself a κ-sensitive and κ-bounded conditional assessment index.

If in addition α is κ-strictly increasing, then (1.20) *is upper semicontinuous, and the unique κ-conditional certainty equivalent of α.*

Remark 1.20. Relation (1.22) shows that C and α reproduce the same ranking, so they are equivalent in this sense. Note that the functional defined in (1.20) satisfies the following property

$$C\left(C\left(X\right)\kappa\right) = C\left(X\right), \quad X \in \mathcal{X},$$

which means that C is a certainty equivalent of itself.

Proof. Let C be defined as in (1.20). Consequently, (1.21) implies that C takes values in L^0. Next we will show that C satisfies (1.19). By locality of α the set $\mathcal{C}(X) := \{m \in L^0 : \alpha(m\kappa) \geq \alpha(X)\}$ is downward directed. Hence, there exits a decreasing sequence $(m_n) \subseteq \mathcal{C}(X)$ converging to $C(X)$ P-almost surely. Upper semicontinuity of α implies

$$\alpha\left(C\left(X\right)\kappa\right) = \alpha\left(\lim_n m_n\kappa\right) \geq \operatorname{ess}\limsup_n \alpha\left(m_n\kappa\right) \geq \alpha(X). \tag{1.23}$$

Suppose now that $\alpha\left(C\left(X\right)\kappa\right) > \alpha\left(X\right)$ on some $A \in \mathscr{G}_+$. By κ-sensitivity of α it follows that $\alpha((C(X) - \varepsilon)\kappa) \geq \alpha(X)$ on A, for some $\varepsilon > 0$ on A. Choose $\varepsilon = 0$ on A^c, and by locality of α and (1.23), we get that $\alpha(C(X)\kappa - \varepsilon) \geq \alpha(X)$. Hence, $C(X) - \varepsilon \in \mathcal{C}(X)$, so that $C(X) - \varepsilon \geq C(X)$, which is a contradiction. Next, let us prove that C is local. By the definition of C, and locality of α, we have

$$\begin{aligned}
&C(1_A X + 1_{A^c} Y)\\
&= \operatorname{ess\,inf}\left\{m \in L^0 : \alpha(m\kappa) \geq \alpha(1_A X + 1_{A^c} Y)\right\}\\
&= \operatorname{ess\,inf}\left\{m \in L^0 : 1_A\alpha(m\kappa) + 1_{A^c}\alpha(m\kappa) \geq 1_A\alpha(X) + 1_{A^c}\alpha(Y)\right\}\\
&= \operatorname{ess\,inf}\{1_A m_1 + 1_{A^c} m_2 \in L^0 : 1_A\alpha\left((1_A m_1 + 1_{A^c} n_1)\kappa\right) \geq 1_A\alpha(X),\\
&\qquad\qquad 1_{A^c}\alpha\left((1_A n_2 + 1_{A^c} m_2)\kappa\right) \geq 1_{A^c}\alpha(Y),\ \text{where } n_1, n_2 \in L^0\}\\
&= 1_A \operatorname{ess\,inf}\{m_1 \in L^0 : 1_A\alpha\left((1_A m_1 + 1_{A^c} n_1)\kappa\right) \geq 1_A\alpha(X), n_1 \in L^0\}\\
&\quad + 1_{A^c} \operatorname{ess\,inf}\left\{m_2 \in L^0 : 1_{A^c}\alpha\left((1_A n_2 + 1_{A^c} m_2)\kappa\right) \geq 1_{A^c}\alpha(Y), n_2 \in L^0\right\}\\
&= 1_A \operatorname{ess\,inf}\left\{1_A m_1 + 1_{A^c} n_1 \in L^0 : 1_A\alpha\left((1_A m_1 + 1_{A^c} n_1)\kappa\right) \geq 1_A\alpha(X)\right\}\\
&\quad + 1_{A^c} \operatorname{ess\,inf}\{1_{A^c} m_2 + 1_A n_2 \in L^0 : 1_{A^c}\alpha\left((1_A n_2 + 1_{A^c} m_2)\kappa\right) \geq 1_{A^c}\alpha(Y)\}\\
&= 1_A C(X) + 1_{A^c} C(Y)
\end{aligned}$$

where in the fourth equality we used the κ-boundedness assumption to ensure the existence of $n_1, n_2 \in L^0$, such that $1_{A^c}\alpha(1_a m_1 + 1_{A^c} n_1)\kappa) \geq 1_{A^c}\alpha(X)$ and $1_A\alpha((1_A n_2 + 1_{A^c} m_2)\kappa) \geq 1_A\alpha(Y)$. Hence, C is local. Thus, C is a κ-conditional certainty equivalent.

Next, we will show that (1.22) is fulfilled. Clearly, $\alpha(X) \geq \alpha(Y)$ implies $C(X) \geq C(Y)$. Suppose that $\alpha(X) \geq \alpha(Y)$, and $\alpha(X) > \alpha(Y)$ on some $A \in \mathscr{G}_+$. Since C is a κ-conditional certainty equivalent of α, it follows that $\alpha(C(X)\kappa) > \alpha(Y)$ on A. By similar arguments as above, since α is κ-sensitive there exists $\varepsilon \in L^0_+$ with $\varepsilon > 0$ on A, and $\varepsilon = 0$ on A^c, such that $\alpha((C(X) - \varepsilon)\kappa) \geq \alpha(Y)$. Hence, $C(X) - \varepsilon \in \mathcal{C}(Y)$, and thus

$C(X) - \varepsilon \geq C(Y)$, which implies that $C(X) > C(Y)$ on A. Thus (1.22) is established.

Note that by means of relation (1.22), α and C define the same conditional preference order on $\mathcal{X}$. Thus, C is itself a conditional assessment index.[13] Also by (1.22) we conclude that α being κ-bounded implies that C is κ-bounded. Next we will show that C is κ-sensitive. Choose $m \in L^0$ and $X \in \mathcal{X}$ such that $C(m\kappa) > C(X)$ on some set $A \in \mathscr{G}$. Using locality of α and C, and by (1.22), it follows that $\alpha(m\kappa) > \alpha(X)$ on A. Hence, by κ-sensitivity of α, there exists $\varepsilon \in L^0_+$ with $\varepsilon > 0$ on A such that $\alpha((m-\varepsilon)\kappa) \geq \alpha(X)$ on A. Again, using locality and (1.22), we conclude that $C((m-\varepsilon)\kappa) \geq C(X)$ on A. This shows that C is κ-sensitive.

Let us assume that α in addition is κ-strictly increasing. We claim that $C(m\kappa) = m$, $m \in L^0$. Indeed, by definition 1.20, we have that $C(m\kappa) \leq m$. Suppose that $C(m\kappa) < m$ on some set $A \in \mathscr{G}_+$. Since α is κ-strictly increasing, it follows that $\alpha\left(C(m\kappa)\right) < \alpha(m\kappa)$ on A. However, $\alpha(C(m\kappa)) = \alpha(m\kappa)$ which is a contradiction. Next we will show that any κ-certainty certainty equivalent $\widetilde{C}$ of α is equal to C. Given $X \in \mathcal{X}$, we note that $\widetilde{C}(X) \in \mathcal{C}(X)$, and hence $C(X) \leq \widetilde{C}(X)$. Suppose that $C(X) < \widetilde{C}(X)$ on some A. Since α is κ-strictly increasing and local, it follows that $\alpha(X) = \alpha(C(X)\kappa) < \alpha(\widetilde{C}(X)\kappa) = \alpha(X)$ on A which is a contradiction. Thus $\widetilde{C} = C$. Finally, it remains to show that C is upper semicontinuous. For a given $m \in L^0$, using the statements proved above, we deduce that

$$\{X \in \mathcal{X} : C(X) \geq m\} = \{X \in \mathcal{X} : C(X) \geq C(m\kappa)\} = \{X \in \mathcal{X} : \alpha(X) \geq \alpha(m\kappa)\}.$$

The latter set is closed since α is upper semicontinuous, and hence, the upper level sets of C are also closed, and thus C is upper semicontinuous. □

Remark 1.21. Note that if α is a κ-bounded and κ-cash additive acceptability index, then, up to a translation by $\alpha(0)$, α is a certainty equivalent of itself. In other terms $C(X) := \alpha(X) - \alpha(0)$ is a certainty equivalent of α. Indeed, κ-boundedness and κ-cash additive imply that α only takes values in L^0, and thus C also takes values only in L^0. Moreover, since $\alpha(m\kappa) = \alpha(0) + m$, we have that $\alpha(C(X)\kappa) = \alpha(0) + C(X) = \alpha(X)$.

1.3 Assessment Indices for Stochastic Processes

We will now apply the theory developed in Section 1.2 to study of assessment indices for discrete time, real valued random processes.

[13] Both monotonicity and quasiconcavity of C follow from corresponding properties of α and relation (1.22).

1.3.1 Conditional Assessment Indices for Stochastic Processes

In this section we follow the approach and notations for stochastic processes introduced by Acciaio et al. [2]. Given a time horizon $T \in \mathbb{N}$, let $(\Omega, \mathscr{F}, P)$ be a probability space with a filtration $(\mathscr{F}_s)$ where s is in $\{0, \dots, T\}$. Given $t \in \{0, \dots, T\}$, we denote by $\mathscr{O}^t$ the optional σ-algebra up to time t on the product space $\widetilde{\Omega} := \Omega \times \{0, \dots, T\}$, which is equal to

$$\mathscr{O}^t = \sigma\left(\{A_s \times \{s\}, A_t \times \{t, \dots, T\} : s < t, A_s \in \mathscr{F}_s \text{ and } A_t \in \mathscr{F}_t\}\right). \tag{1.24}$$

We define $\mathscr{O} := \mathscr{O}^T$. On $\widetilde{\Omega}$ we denote by $\widetilde{P}$ a probability measure, which is defined by the expectation

$$E_{\widetilde{P}}[X] := E_P\left[\sum_{s=0}^{T} X_s \mu_s\right],$$

where μ is some adapted process such that $\sum_{s=0}^{T} \mu_s = 1$ and $\mu_s > 0$. We shall sometimes write $\widetilde{P} = P \otimes \mu$.

Note that a random variable X belongs to $L^0(\mathscr{O}^t)$ if, and only if, seen as a process $X = (X_s)$, it is $(\mathscr{F}_s)$-adapted up to time t and constant afterwards.[14] In particular, any $X \in L^0(\mathscr{O})$, seen as a process, is $(\mathscr{F}_s)$-adapted and it is clear that $L^0(\mathscr{O}^{t_1}) \subseteq L^0(\mathscr{O}^{t_2})$ for any $t_1, t_2 \in \{0, \dots, T\}$ with $t_1 \leq t_2$.

For any $X \in L^0(\mathscr{O})$, we denote by $\Delta X_s := (X_s - X_{s-1})$, with the convention $X_{-1} = 0$, so that $X_s = \sum_{k=0}^{s} \Delta X_s$.

Remark 1.22. In what follows a process $X \in L^0(\mathscr{O})$ will be interpreted either as a discounted cumulative cash flow (discounted cumulative dividend) process, or as a discounted cash flow process (discounted dividend process). If X is a discounted cumulative cash flow, then ΔX represents the discounted dividend process.

From now through the end of this subsection we fix $t \in \{0, 1, \dots, T\}$. For $q \in [1, +\infty]$, we denote by $\widetilde{\mathcal{M}}_{q,t}$, the set of probability measures $\widetilde{Q}$ on $\mathscr{O}$ absolutely continuous with respect to $\widetilde{P}$, such that $d\widetilde{Q}/d\widetilde{P} \in L^q(\mathscr{O})$ and $\widetilde{Q} = \widetilde{P}$ on $\mathscr{O}^t$. In case $q = 1$, and if no confusion arises, we will drop q from the notations. Similarly, we denote by $\mathcal{M}_t$ the set of probability measures Q on $\mathscr{F}_T$ absolutely continuous with respect to P, such that $dQ/dP \in L^1(\mathscr{F}_T)$ and $Q = P$ on $\mathscr{F}_t$. Given $Q \in \mathcal{M}_t$ we denote by $\Gamma_t(Q)$ and $\mathcal{D}_t(Q)$ the set of optional random measures and predictable discounting processes[15] from time

[14] By "constant afterwards" we mean that $X_s = X_t$ for $s \geq t$.

[15] It is important to stress that process D does not represent a financial discount factor. For the meaning and the role of this process we refer to Theorem 1.26.

t respectively, that is

$$\Gamma_t(Q) := \left\{ (\gamma_s) \in L^0_+(\mathscr{O}) : \gamma_0 = \ldots = \gamma_{t-1} = 0 \text{ and } \sum_{s=t}^{T} \gamma_s = 1,\ Q\text{-almost surely} \right\},$$
$$\mathcal{D}_t(Q) := \big\{ (D_t) \in L^0_+(\mathscr{O}) : D_0 = \ldots = D_t = 1,\ Q\text{-almost surely}, \quad D \text{ is predictable and decreasing} \big\}.$$

Lemma 1.23. *Let $Q \in \mathcal{M}_t$. There exists a one-to-one relation between $\gamma \in \Gamma_t(Q)$ and $D \in \mathcal{D}_t(Q)$ given by*

$$D_0 = 1, \quad \textit{and} \quad D_s = 1 - \sum_{k=0}^{s-1} \gamma_k, \quad \textit{for} \quad 0 < s \leq T,$$

$$\gamma_s = D_s - D_{s+1}, \quad \textit{for} \quad 0 \leq s < T \quad \textit{and} \quad \gamma_T = 1 - \sum_{k=0}^{T-1} \gamma_k = D_T. \tag{1.25}$$

Furthermore, for any $X \in L^0(\mathscr{O})$, it holds

$$\langle \gamma, X \rangle_t := \sum_{s=t}^{T} \gamma_s X_s = X_t + \sum_{s=t+1}^{T} D_s \Delta X_s =: (D \bullet X)_t \tag{1.26}$$

with the convention that $D_{T+1} = 0$.

Finally, $\widetilde{Q} \in \widetilde{\mathcal{M}}_t$ if and only if there exists $Q \in \mathcal{M}_t$ and $\gamma \in \Gamma_t(Q)$ or the corresponding $D \in \mathcal{D}_t(Q)$ such that[16] *$\widetilde{Q} = Q \otimes \gamma$ or $\widetilde{Q} = Q \otimes D$.*

This was proven in [2]. Note that the additional term X_t in (1.26) of the integration by part is missing in [2]. Next we define the sets[17]

$$\mathcal{M} \otimes_t \mathcal{D} := \{ Q \otimes D : Q \in \mathcal{M}_1 \text{ and } D \in \mathcal{D}_t(Q) \};$$
$$\mathcal{M} \otimes_{q,t} \mathcal{D} := \left\{ Q \otimes D : Q \in \mathcal{M}_t, D \in \mathcal{D}_t(Q), \text{ and } Q \otimes D \in \widetilde{\mathcal{M}}_{q,t} \right\}, \quad q \in (1, +\infty].$$

Remark 1.24. By means of Lemma 1.23, it holds $\widetilde{Q} \in \widetilde{\mathcal{M}}_{q,t}$ if and only if $\widetilde{Q} = Q \otimes D \in \mathcal{M} \otimes_{q,t} \mathcal{D}$, or $\widetilde{Q} = Q \otimes \gamma \in \mathcal{M} \otimes_{q,t} \Gamma$, $q \in [1, \infty]$.

[16] Where $Q \otimes \gamma$ has to be understood as the product measure with density $(Z_t \frac{\gamma_t}{\mu_t})$, whereby $Z_t = dQ/dP_{|\mathscr{F}_t}$ and $Q \otimes D$ is the product measure with density $(Z_t \frac{(D_t - D_{t+1})}{\mu_t})$.

[17] Analogously, we define the sets $\mathcal{M} \otimes_t \Gamma$, and $\mathcal{M} \otimes_{q,t} \Gamma, q \in (1, \infty]$.

Following [61] we define the conditional p-norm

$$\|X\|_{t,p} := \begin{cases} E_{\widetilde{P}}\left[|X|^p \;\middle|\; \mathscr{O}^t\right]^{1/p}, & \text{if } p < \infty \\ \\ \operatorname{ess\,inf}\left\{\xi \in L^0(\mathscr{O}^t) : |X| \leq \xi\right\}, & \text{if } p = \infty, \end{cases}$$

on the basis of which we define the spaces

$$L^{t,p}\left(\mathscr{O}\right) := \left\{X \in L^0\left(\mathscr{O}\right) : \|X\|_{t,p} \in L^0\left(\mathscr{O}^t\right)\right\}.$$

By means of [61, Proposition 4.4], it holds that

$$L^{t,p}\left(\mathscr{O}\right) = L^0\left(\mathscr{O}^t\right) L^p\left(\mathscr{O}\right), \quad 1 \leq p \leq \infty.$$

It is shown in [61] that $(L^{t,p}(\mathscr{O}), \|\cdot\|_{t,p})$, with the order of almost sure dominance, is an $L^0\left(\mathscr{O}^t\right)$–normed module lattice. For a fixed $0 \leq t \leq T$ and $1 \leq p \leq \infty$, we let $\mathcal{X} = L^{t,p}(\mathscr{O})$. We equip $\mathcal{X} = L^{t,p}(\mathscr{O})$ with the $\|\cdot\|_{t,p}$-topology for $1 \leq p < \infty$, or the conditional weak*-topology $\sigma(\mathcal{X}, L^{t,1}(\mathscr{O}))$ if $p = \infty$.

We say that a functional $\alpha : \mathcal{X} \to \bar{L}^0(\mathscr{O}^t)$ is monotone if $\alpha(X) \geq \alpha(Y)$ whenever $X \geq Y$ $\widetilde{P}$-almost surely[18].

In the case $\mathcal{X} = L^{t,\infty}(\mathscr{O})$, upper semicontinuity with respect to $\sigma(L^{t,\infty}(\mathscr{O}), L^{t,1}(\mathscr{O}))$ can be characterized as follows. A function $\alpha : L^{t,\infty}(\mathscr{O}) \to \bar{L}^0(\mathscr{O}^t)$ is said to fulfill the Fatou property if for every sequence $(X_n)_{n\in\mathbb{N}}$ with $X_n \downarrow X$ it holds that $\alpha(X_n) \downarrow \alpha(X)$.

Lemma 1.25. *A local, quasiconcave function $\alpha : L^{t,\infty}(\mathscr{O}) \to \bar{L}^0(\mathscr{O}^t)$ fulfilling the Fatou property is $\sigma(L^{t,\infty}(\mathscr{O}), L^{t,1}(\mathscr{O}))$ upper semicontinuous.*

Proof. We first show that a function $\alpha : L^{t,\infty}(\mathscr{O}) \to \bar{L}^0(\mathscr{O}^t)$ fulfills the Fatou property if and only if for every bounded sequence $(X_n)_{n\in\mathbb{N}} \subseteq L^{t,\infty}(\mathscr{O})$ converging P-almost surely to X it holds that

$$\alpha(X) \geq \limsup \alpha(X_n).$$

First, we show the implication of the Fatou property. To this end, consider a sequence $X_n \downarrow X$ implying $X_n \geq X$ for every $n \in \mathbb{N}$. By monotonicity of α it follows that $\alpha(X_n) \geq \alpha(X)$ for every $n \in \mathbb{N}$. Hence, we obtain that $\alpha(X) \geq \limsup \alpha(X_n) \geq \alpha(X)$, showing $\alpha(X_n) \downarrow \alpha(X)$. Reversely, suppose that α fulfills the Fatou property. Consider a sequence $(X_n)_{n\in\mathbb{N}}$ converging to some X. Defining $Y_m := \operatorname{ess\,sup}_{n\geq m} X_n$, $m \in \mathbb{N}$, it holds that $Y_n \geq X_n$ for every $n \in \mathbb{N}$. Moreover, since $Y_n \downarrow X$ it follows that $\alpha(Y_n) \downarrow \alpha(X)$. Hence, $\alpha(X) = \lim \alpha(Y_n) \geq \limsup X_n$ showing the Fatou property.

It remains to show that the upper level set $\mathcal{A}^m = \{X \in L^{t,\infty}(\mathscr{O}) : \alpha(X) \geq m\}$,

[18] The monotonicity in this case coincides with the monotonicity with respect to the cone $\mathcal{K} = \{X : X \geq 0\}$.

$m \in L^0(\mathscr{O}^t)$, is $\sigma(L^{t,\infty}(\mathscr{O}), L^{t,1}(\mathscr{O}))$-closed. Since α is quasiconcave, it holds that $\mathcal{A}^m$ is $L^0(\mathscr{O}^t)$-convex. By the conditional Krein-Šmulian theorem, the proof of which can be found in chapter 4, it is hence sufficient to show $\mathcal{A}^m_r := \mathcal{A}^m \cap \{X \in L^{t,\infty}(\mathscr{O}) : \|X\| \leq r\}$ is $\sigma(L^{t,\infty}(\mathscr{O}), L^{t,1}(\mathscr{O}))$-closed for every $r \in L^0(\mathscr{O}^t)$. To this end, consider a sequence $(X_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}^m_r$ converging to X in $L^{t,1}(\mathscr{O})$. Hence, we may extract subsequence $(X_{n_m})_{m\in\mathbb{N}}$ converging P-almost surely to X and apply the Fatou property to conclude $\alpha(X) \geq \limsup \alpha(X_{n_m}) \geq m$. Thus, X is also in $\mathcal{A}^m_r$, which shows that $\mathcal{A}^m_r$ is $L^{t,1}(\mathscr{O})$-closed. However, as a consequence of the conditional Hahn-Banach theorem, which is proven in chapter 4, an $L^0(\mathscr{O}^t)$-convex, $L^{t,1}(\mathscr{O})$-closed set is also $\sigma(L^{t,\infty}(\mathscr{O}), L^{t,1}(\mathscr{O}))$-closed. This finishes the proof. $\square$

Theorem 1.26. *Let $\alpha : \mathcal{X} \to \bar{L}^0(\mathscr{O}^t)$ be an upper semicontinuous conditional assessment index. Then α has a robust representation of the form*

$$\alpha(X) = \operatorname*{ess\,inf}_{\widetilde{Q}\in\widetilde{\mathcal{M}}_{q,t}} R\left(\widetilde{Q}, E_{\widetilde{Q}}\left[X \mid \mathscr{O}^t\right]\right), \tag{1.27}$$

for a unique minimal risk function $R : \widetilde{\mathcal{M}}_{q,t} \times \bar{L}^0(\mathscr{O}^t) \to \bar{L}^0(\mathscr{O}^t)$.

This robust representation can be written in the following form

$$\alpha_s(X) = f_s(X_s), \quad s \leq t-1, \tag{1.28}$$

and,

$$\alpha_s(X) = \alpha_t(X) = \operatorname*{ess\,inf}_{Q\otimes\gamma\in\mathcal{M}\otimes_{q,t}\Gamma} R'_t\left(Q\otimes\gamma, E_Q\left[\sum_{k=t}^{T}\gamma_k X_k \;\middle|\; \mathscr{F}_t\right]\right) \tag{1.29}$$

$$= \operatorname*{ess\,inf}_{Q\otimes D\in\mathcal{M}\otimes_{q,t}\mathcal{D}} R'_t\left(Q\otimes D, X_t + E_Q\left[\sum_{k=t+1}^{T} D_k\Delta X_k \;\middle|\; \mathscr{F}_t\right]\right), \quad s \geq t, \tag{1.30}$$

for an unique right-continuous increasing functions $f_s : L^0_s \to \bar{L}^0(\mathscr{F}_s)$ and minimal risk functions $R'_t : \mathcal{M}\otimes_{q,t}\Gamma \times \bar{L}^0(\mathscr{F}_t) \to \bar{L}^0(\mathscr{F}_t)$.

Remark 1.27. From the financial point of view, the representation (1.29) is meaningful if X is a discounted cash flow (discounted dividend process), and the representation (1.30) is meaningful if X is a discounted cumulative cash flow (discounted cumulative dividend process).

Proof. Since α is monotone with respect to cumulative cash flows, it holds $X \succcurlyeq Y$ if and only if $X - Y \in \mathcal{K} := \{U \in \mathcal{X} : U \geq 0\}$ and so $\mathcal{K}^\circ = \{Z \in L^{t,q}(\mathscr{O}) : Z \geq 0\}$. We will make use of the normalized polar cone $\mathcal{K}^\circ_1 := \{Z \in L^{t,q}(\mathscr{O}) : Z \geq 0 \text{ and } E_{\widetilde{P}}[Z \mid \mathscr{O}^t] = 1\}$, which can be identified with $\widetilde{\mathcal{M}}_{q,t}$. Applying Theorem 1.12 and Remark 1.13, there exists a unique minimal conditional risk function $R : \widetilde{\mathcal{M}}_{q,t} \times \bar{L}^0(\mathscr{O}^t) \to \bar{L}^0(\mathscr{O}^t)$ such that the representation (1.27) holds true.

To show the second claim of the theorem assume first that $p = \infty$. First note that

$$E_{\widetilde{Q}}\left[X \;\middle|\; \mathscr{O}^t\right] = \left(X'_0, \ldots, X'_{t-1}, E_Q\left[\langle\gamma, X\rangle_t \;\middle|\; \mathscr{F}_t\right], \ldots, E_Q\left[\langle\gamma, X\rangle_t \;\middle|\; \mathscr{F}_t\right]\right), \qquad (1.31)$$

for all $X \in L^{t,\infty}(\mathscr{O})$ and all $\widetilde{Q} = Q \otimes \gamma$, where $Q \in \mathcal{M}_t$ and $\gamma \in \Gamma_t(Q)$, and where X' is any element of $L(\mathscr{O})$. Indeed, suppose that $X \in L^{t,\infty}(\mathscr{O})$, and denote by Y the random variable on the right hand side of (1.31). Let $A = (A_0, A_1, \ldots, A_t, A_t, \ldots, A_t)$ such that $A_s \in \mathscr{F}_s$ for any $s \le t$. Then,

$$\begin{aligned}
E_{\tilde{Q}}\left[X 1_A\right] &= \sum_{s=0}^{t-1} E_Q\left[X_s 1_{A_s}\gamma_s\right] + \sum_{s=t}^{T} E_Q\left[X_s 1_{A_t}\gamma_s\right] \\
&= 0 + E_Q\left[E_Q\left[\sum_{s=t}^{T} X_s\gamma_s \;\middle|\; \mathscr{F}_t\right] 1_{A_t}\right] \\
&= E_{\tilde{Q}}\left[Y 1_A\right],
\end{aligned}$$

and hence (1.31) is proved. For convenience, we will take $X' = X$ in what follows.

By Remark 1.24, $\widetilde{Q} \in \widetilde{\mathcal{M}}_t$ if and only if $\widetilde{Q} = Q \otimes \gamma$ for $Q \in \mathcal{M}_t$ and $\gamma \in \Gamma_t(Q)$. For $s \le t-1$ we use locality for $A = \Omega \times \{s\} \in \mathscr{O}^t$ which yields $1_{\{s\}}\alpha(1_{\{s\}}X) = 1_{\{s\}}\alpha(X)$ since $1_A = 1_{\{s\}}$. Thus, $\alpha_s(X) = \alpha_s(0, \ldots, X_s, 0, \ldots) =: \alpha_s(X_s)$. Since[19] $1_{\{s\}}(Q \otimes \gamma) = 1_{\{s\}}$, for any $Q \in \mathcal{M}_t, \gamma \in \Gamma_t(Q)$, and using locality of R and (1.31), we deduce that

$$\alpha_s(X) = \alpha_s(X_s) = \operatorname*{ess\,inf}_{Q\otimes\gamma\in\mathcal{M}\otimes_t\Gamma} R_s(1_{\{s\}}(Q\otimes\gamma), (0, \ldots, X_s, 0, \ldots)) =: f_s(X_s), \quad s \le t-1,$$

and thus (1.28) is established. In the case $s \ge t$ we apply locality to the set $\Omega \times \{t, \ldots, T\}$. Hence, we see that $\alpha_s(X)$ is equal to $\alpha_t(X)$ for all $s \ge t$ and using (1.31) we get

$$\alpha_t(X) = \operatorname*{ess\,inf}_{Q\otimes\gamma\in\mathcal{M}\otimes_t\Gamma} R'_t\left(Q\otimes\gamma, E_Q\left[\langle\gamma, X\rangle_t \;\middle|\; \mathscr{F}_t\right]\right),$$

where

$$R'_t\left(Q\otimes\gamma, s_t\right) := R_t\left(Q\otimes\gamma, (0, \ldots, 0, s_t, \ldots, s_t)\right), \quad s_t \in \bar{L}^0(\mathscr{F}_t),$$

is a uniquely determined risk function. This proves the representation (1.29). By Lemma 1.23 and (1.29), the represetnation (1.30) follows immediately.

As for the case $1 \le p < +\infty$, in view of Remark 1.24, and proceeding analogously as above, we conclude that (1.29) and (1.30) are satisfied. □

Remark 1.28. It is in place here to remark that the assessment index α considered in

[19] By $1_{\{s\}}(Q\otimes\gamma)$ we naturally mean the density of $Q\otimes\gamma$ with respect to $\widetilde{P}$ at time s.

this subsection corresponded to the fixed t. It would be then appropriate to denote it as, say, $\alpha^t = (\alpha^t_0, \dots, \alpha^t_T)$. We would then refer to the collection $\{\alpha^t,\ t = 0, 1, \dots, T\}$ as to dynamic assessment index.

1.3.2 Path Dependent Dynamic Assessment Indices

Throughout this section we interpret X as the discounted cumulative cash-flow.

It is seen from representation (1.29) that α^t_t (compare Remark 1.28) only assesses the future of the process X, that is it only assesses $X_t, \dots, X_T$, while $\alpha^t_s,\ s < t$, is just a function of X_s. This is a drawback since the past evolution of X is not taken into account when assessing X at time t via α^t_t, which for some applications may be an unwanted feature.

In this section we propose an alternative approach, which assess X at time t accounting for the path evolution of X time t.

Given $0 \le s \le \tilde{s} \le T$, we denote by $1_{[s,\tilde{s}]}$ a process, such that $1_{[s,\tilde{s}]}(u) = 1$ for $s \le u \le \tilde{s}$, and $1_{[s,\tilde{s}]}(u) = 0$ otherwise. Accordingly, we use the notation $X_{[s,\tilde{s}]}$ for the random vector $X1_{[s,\tilde{s}]} = (0, \dots, 0, X_s, \dots, X_{\tilde{s}}, 0, \dots, 0)$. Process X stopped at time t is written as X^t, that is $X^t = X_{\cdot \wedge t}$. We recall the definition of the space $L^0(\mathscr{O}^t)$ (compare (1.24)), and we define

$$L^0(\mathscr{O}_{[s,\tilde{s}]}) := \left\{ X_{[s,\tilde{s}]} : X \in L^0(\mathscr{O}) \right\}.$$

We remark that $\mathscr{O}_{[s,\tilde{s}]}$ is understood as the optional σ-algebra generated by processes $X_{[s,\tilde{s}]}$.

Hence, for a fixed t we may decompose any process $X \in L^0(\mathscr{O})$ as follows

$$X = X_{[0,t-1]} + X_{[t,T]} = X^{t-1} + (X_{[t,T]} - X_{t-1}1_{[t,T]}),$$

where $X_{[0,t-1]} \in L^0(\mathscr{O}_{[0,t-1]})$, $X_{[t,T]} \in L^0(\mathscr{O}_{[t,T]})$ and $X^{t-1} \in L^0(\mathscr{O}^{t-1})$.

It is evident that $L^0(\mathscr{O}_{[t,T]})$ is an $L^0(\mathscr{F}_t)$-module[20]. We further define

$$\hat{\mathcal{M}}_{q,t} := \left\{ \hat{Q} : \hat{Q} \text{ measure on } \Omega \times \{0, \dots, T\}, \hat{Q} \ll \hat{P} := P \otimes \mu, d\hat{Q}/d\hat{P} \in L^q(\mathscr{O}_{[t,T]}) \right\},$$

where μ is a measure on $\{t, \dots, T\}$ such that $\mu_s > 0$ for every $s \in \{t, \dots, T\}$.

We further denote

$$\hat{\mathcal{M}} \otimes_{q,t} \hat{\mathcal{D}} := \left\{ Q \otimes D : Q \in \mathcal{M}_1, D \in \mathcal{D}_t(Q), \text{ and } Q \otimes D \in \hat{\mathcal{M}}_{q,t} \right\}.$$

Remark 1.29. In this setting, let $\hat{Q} \in \hat{\mathcal{M}}_{1,t}$, and denote by $\Lambda = d\hat{Q}/d\hat{P} \in L^1(\mathscr{O}_{[t,T]})$. It holds that $U = (U_s)_{s=t}^T$, where $U_s = E_P[\sum_{k=s}^T \Lambda_k \mu_k \,|\, \mathscr{F}_s]$ for $s \in \{t, \dots, T\}$, is a

[20] For the multiplication $\lambda X_{[t,T]} = (0, \dots, 0, \lambda X_t, \dots, \lambda X_T)$, $\lambda \in L^0(\mathscr{F}_t)$.

super martingale fulfilling additionally $E_P[U_{t+1}\,|\,\mathscr{F}_t] = U_t = 1$. Hence, using the Itô-Watanbe decomposition $U = ZD$ where D is a predictable decreasing process and Z is a martingale, it follows that $D_t = 1$ and Z_T is a density of a probability measure $Q \in \mathcal{M}_t$. Reciprocally, $\Lambda_k = Z_k(D_k - D_{k+1})/\mu_k$ for every $k = t, \dots, T-1$, and $\Lambda_T = Z_T D_T/\mu_T$, where Z is a martingale and D is a predictable decreasing process with $D_t = 1$, defines a density process for some $\hat{Q} \in \hat{\mathcal{M}}_{1,t}$. Hence, for every $X_{[t,T]} \in L^{t,p}(\mathscr{O}_{[t,T]})$,[21] it follows that

$$
\begin{aligned}
&E_{\hat{Q}}\left[X_{[t,T]}\;\middle|\;\mathscr{F}_t\right]\\
=&E_{\hat{P}}\left[\Lambda X_{[t,T]}\;\middle|\;\mathscr{F}_t\right] = E_P\left[\sum_{k=t}^{T}\Lambda_k X_k \mu_k\;\middle|\;\mathscr{F}_t\right]\\
=&E_Q\left[\sum_{k=t}^{T-1}(D_k - D_{k+1})X_k + D_T X_T\;\middle|\;\mathscr{F}_t\right] = E_Q\left[X_t + \sum_{k=t+1}^{T} D_k \Delta X_k\;\middle|\;\mathscr{F}_t\right].
\end{aligned}
\tag{1.32}
$$

We finally set

$$\mathcal{X}_t^p := \{X \in L^0(\mathscr{O}) : X_{[t,T]} \in L^{t,p}(\mathscr{O}_{[t,T]})\}.$$

Definition 1.30. A function $\alpha : \mathcal{X}_t^p \to \bar{L}^0(\mathscr{F}_t)$ is called an upper semicontinuous path dependent assessment index if for every fixed path $\bar{X} \in L^0(\mathscr{O}^{t-1})$, the function

$$X_{[t,T]} \longmapsto \alpha\left(\bar{X}_{[0,t-1]} + X_{[t,T]}\right), \quad X_{[t,T]} \in L^{t,p}(\mathscr{O}_{[t,T]}),$$

is an upper semicontinuous assessment index.

Theorem 1.31. *Let α be an upper semicontinuous path dependent assessment index. Then it has a robust representation of the form*

$$\alpha(X) = \operatorname*{ess\,inf}_{Q\otimes D \in \hat{\mathcal{M}}\otimes_{q,t}\hat{\mathcal{D}}} R\left(X_{[0,t-1]}; Q\otimes D; E_Q\left[X_t + \sum_{k=t+1}^{T} D_k\Delta X_k\;\middle|\;\mathscr{F}_t\right]\right),$$

for a unique function $R : L^0(\mathscr{O}_{[0,t-1]}) \times \hat{\mathcal{M}} \otimes_{q,t} \hat{\mathcal{D}} \times \bar{L}^0(\mathscr{F}_t) \to \bar{L}^0(\mathscr{F}_t)$ for which $R(X_{[0,t-1]}, \cdot, \cdot) : \hat{\mathcal{M}} \otimes_{q,t} \hat{\mathcal{D}} \times \bar{L}^0(\mathscr{F}_t) \to \bar{L}^0(\mathscr{F}_t)$ is a maximal risk function for every $X_{[0,t-1]} \in L^0(\mathscr{O}_{[0,t-1]})$.

Proof. First, we fix $\bar{X} \in L^0(\mathscr{O}^{t-1})$ and we apply Theorem 1.12 to $\alpha(\bar{X} + \cdot)$ in the fashion analogous the the proof of Theorem 1.26 in order to get the following representation

$$\alpha\left(\bar{X}_{[0,t-1]} + X_{[t,T]}\right) = \operatorname*{ess\,inf}_{\hat{Q}\in\hat{\mathcal{M}}_{q,t}} \bar{R}\left(\bar{X}_{[0,t-1]}, \hat{Q}, E_{\hat{Q}}\left[X_{[t,T]}\,|\,\mathscr{F}_t\right]\right). \tag{1.33}$$

Similarly as in Remark 1.24, we also have that $\hat{Q} \in \hat{\mathcal{M}}_{q,t}$ if and only if $\hat{Q} = Q \otimes D \in \hat{\mathcal{M}} \otimes_{q,t} \hat{\mathcal{D}}$. Hence, using (1.32) in representation (1.33), we conclude the proof. □

[21] In analogy to $L^{t,p}(\mathscr{O}) = L^0(\mathscr{O}^t)L^p(\mathscr{O})$, we have $L^{t,p}(\mathscr{O}_{[t+1,T]}) = L^0(\mathscr{F}_t)L^p(\mathscr{O}_{[t+1,T]})$.

Note that α is no longer local with respect $\mathscr{O}^{t-1}$ on $\Omega \times \{0,\dots,T\}$. Let us now consider the following illustrating example.

Example 1.32. Let us consider a function $\alpha : \mathcal{X}_t^p \to \bar{L}^0(\mathscr{F}_t)$ given by the following formula

$$\alpha(X) = \sum_{k=0}^{t-1} D'_k \Delta X_k + \operatorname*{ess\,inf}_{Q\otimes D \in \hat{\mathcal{M}} \otimes_{q,t} \hat{\mathcal{D}}} R'\left(Q \otimes D, X_t + E_Q\left[\sum_{k=t+1}^{T} D_k \Delta X_k \;\Big|\; \mathcal{F}_t\right]\right)$$

where D' is an adapted process, and $R'(\cdot,\cdot) : \hat{\mathcal{M}} \otimes_{q,t} \hat{\mathcal{D}} \times \bar{L}^0(\mathscr{F}_t) \to \bar{L}^0(\mathscr{F}_t)$ is a maximal risk function. Then, such α is a an upper semicontinuous path dependent assessment index.

The whole process $(D'_0,\dots,D'_{t-1},1,D_{t+1},\dots,D_T)$ may be interpreted as weighing the past and the future of the cash flows, relative to the present time t.

Depending on the specification of D'_k, we get,

- if all $D'_k = 0$, a representation of path independent assessment indices.
- If all $D'_k = 1$, then $\sum_{k=0}^{t-1} \Delta X_k = X_{t-1}$, which means that α depends only on the assessment of the future returns starting at the previous level of wealth X_{t-1}.
- Changing the parameter D'_k in between, one puts more or less weight on the past evolution of returns.

This kind of path dependence indicates how the past evolution of discounted cumulative cash flow may influence the present assessment of the entire investment process. On the one hand, in terms of preferences, such index could provide a model that explains well why in a market experiencing recent period of good performance, the assessment is "optimistic," since distant past bad returns could be discounted more than the recent good ones. On the other hand, such index may provide some guidelines to the regulator to implement contra cyclical measure. Indeed, they could require D' to be dependent on the past returns, in way that puts more weight in times of good returns and less weight in times of bad returns. Such a weighting factor reflecting this feature could take the form

$$D'_k = \exp\left(0.08 - \frac{\Delta X_k}{X_k}\right),$$

where 8% were a reasonable annual return for a banking institution.

Remark 1.33. Similarly as in Remark 1.28 we observe that the assessment index α considered in this subsection corresponded to the fixed t. It would be then appropriate to denote it as, say, α_t. We would then refer to the collection $\{\alpha_t,\ t = 0,1,\dots,T\}$ as to dynamic path dependent assessment index.

1.4 Dynamically Consistent Assessment Indices

In this section we discuss the key notion of dynamic consistency with regard to assessment indices. Here, we only focus on the so called strong dynamic consistency for path dependent assessment indices. For other notions of time consistency compare Acciaio et al. [2], Acciaio and Penner [1] and references therein, with regard to dynamic risk measures, and we refer to Bielecki et al. [12] and Biagini and Bion-Nadal [9] with regard to acceptability indices.

We consider a dynamic path dependent assessment index $\alpha = \{\alpha_t,\ t = 0, \dots, T\}$ (compare Remark 1.33).

Definition 1.34. We say that α is strongly time consistent if for any $X, Y \in \mathcal{X}_t^p$ and t such that $X_{[0,t]} = Y_{[0,t]}$ the following implication is true

$$\alpha_{t+1}(X) \geq \alpha_{t+1}(Y) \quad \text{implies} \quad \alpha_t(X) \geq \alpha_t(Y).$$

Remark 1.35. One needs to observe that the notion of strong time consistency seems to be inappropriate in the case of scale invariant assessment indices. Indeed, let α be scale invariant and strongly time consistent. Assume that $X_{[0,t]}Y_{[0,t]} \geq 0$ and $\alpha_{t+1}(X) \geq \alpha_{t+1}(Y)$. Then, there exists $\lambda \in L^0_{++}(\mathscr{O}^t)$ such that $\lambda X_{[0,t]} = Y_{[0,t]}$, and in view of scale invariance of α, we have that $\alpha_t(X) \geq \alpha_t(Y)$. Thus the condition $X_{[0,t]} = Y_{[0,t]}$ appears to be irrelevant for the strong time consistency in this case, which is unreasonable from the risk management point of view. Consequently, a different notion of time consistency is needed in case of scale invariant assessment indices. One such possible notion was introduced and studied in [12].

Moreover, as shown below, the strong time consistency is strongly related to existence of a certainty equivalent, which fails to exists (compare Remark 1.17) for scale invariant assessment indices.

In order to derive a version of the so called Bellman principle, some additional assumptions have to be done. We suppose throughout this section that $X_{[t,T]} \mapsto \alpha_t(X_{[0,t-1]} + X_{[t,T]})$ fulfills the assumptions of Proposition 1.19 with the boundedness assumption given for $m_1, m_2 \in L^p(\mathscr{F}_t)$ rather than $L^0(\mathscr{F}_t)$.

Let us define a family of functionals $C_t : \mathcal{X}_t^p \to \bar{L}^0(\mathscr{F}_t)$ for $t = 0, 1, \dots, T$, by

$$C_t(X) := \operatorname{ess\,inf} \left\{ m_t \in L^p(\mathscr{F}_t) : \alpha_t(X_{[0,t-1]} + m_t 1_{[t,T]}) \geq \alpha_t(X) \right\}.$$

According to Proposition 1.19, for each t, $X_{[t,T]} \mapsto C_t(X_{[0,t-1]} + X_{[t,T]})$ is an upper semicontinuous (path dependent) assessment index taking values into $L^p(\mathscr{F}_t)$ such that

$$\alpha_t(X) \geq \alpha_t(Y) \quad \text{if, and only if} \quad C_t(X) \geq C_t(Y).$$

In particular $C_t(X_{[0,t-1]} + C_t(X)1_{[t,T]}) = C_t(X)$. In addition, the family α is strongly time consistent if and only if the family $C := (C_t)$ is strongly time consistent.

With this at hand, we may formulate the following version of the celebrated Bellman principle.

Proposition 1.36. *Under the assumptions adopted in this section, if α is strongly time consistent, the corresponding family C of certainty equivalents satisfies, for each $t = 0, \dots, T-1$,*

$$C_t(X) = C_t(X_{[0,t]} + C_{t+1}(X)1_{[t+1,T]}), \quad X \in \mathcal{X}_t^p. \tag{1.34}$$

Proof. Since C_{t+1} is a certainty equivalent, it follows that $C_{t+1}(X) = C_{t+1}(X_{[0,t]} + C_{t+1}(X)1_{[t+1,T]})$. By means of the boundedness assumption, $C_{t+1}(X) \in L^p(\mathscr{F}_t)$, and so defining $Y = X_{[0,t]} + C_{t+1}(X)1_{[t+1,T]}$, it follows that $Y \in \mathcal{X}_t^p$ and $Y_{[0,t]} = X_{[0,t]}$. Thus, the strong time consistency applied to C yields (1.34). □

From now on, we consider certainty equivalent corresponding to assessment indices fulfilling the conditions from Proposition 1.36. Note that for $X_{[0,t]} \in L^0(\mathscr{O}^t)$, the function $C_t : L^p_{t+1}(\mathscr{F}_t) \to L^p(\mathscr{F}_t)$, $Y \mapsto C_t(X_{[0,t]} + Y1_{[t+1,T]})$ is an upper semicontinuous assessment index, and we denote by $R_{t,t+1}$ its corresponding minimal risk function, for which it holds

$$\begin{aligned}
&C_t(X_{[0,t]} + Y1_{[t+1,T]}) \\
&= \operatorname*{ess\,inf}_{Q\otimes D \in \mathcal{MD}_t^{t+1}} R_{t,t+1}\left(X_{[0,t-1]}, Q\otimes D, X_t + E_Q\left[D\left(Y - X_t\right) \;\middle|\; \mathscr{F}_t\right]\right),
\end{aligned}$$

where

$$\mathcal{MD}_t^{t+1} := \left\{Q\otimes D : Q \in \mathcal{M}_t^{t+1}, 0 \le D \le 1 \text{ and } D \text{ is } \mathscr{F}_t\text{-measurable}\right\},$$

whereby $\mathcal{M}_t^{t+1}$ denotes the set of probability measures Q on $\mathscr{F}_{t+1}$ such that $Q \ll P$ and $Q = P$ on $\mathscr{F}_t$. As a convention, we set $\mathcal{MD}_T^{T+1} = \{1\}$ since $C_T(X) = X_T$.

Theorem 1.37. *If $\alpha = (\alpha_t)$ is a strongly time consistent sequence of path dependent assessment indices fulfilling the assumptions of Proposition 1.19, then*

$$C_t(X) = \operatorname*{ess\,inf}_{Q\otimes D \in \mathcal{MD}_t^{t+1}} F_t(Q\otimes D, X); \quad X \in \mathcal{X}_t^p,$$

where

$$\begin{aligned}
&F_t(Q\otimes D, X) = \\
&\operatorname*{ess\,inf}_{\bar{Q}\otimes\bar{D} \in \mathcal{MD}_{t+1}^{t+2}} R_{t,t+1}\left(X_{[0,t-1]}, Q\otimes D, E_Q\left[D\Big(F_{t+1}(\bar{Q}\otimes\bar{D}, X) - X_t\Big) + X_t \;\middle|\; \mathscr{F}_t\right]\right),
\end{aligned}$$

for $t \leq T-1$ *and*

$$F_T(Q \otimes D, X) = X_T, \quad Q \otimes D \in \mathcal{MD}_T^{T+1} = \{1\}.$$

Proof. Let us prove the theorem for $t = T-1, T-2$; the rest of the proof follows by backward recursion. Clearly, $C_T(X) = X_T$. As for $t = T-1$, since $\mathcal{MD}_{T-1} = \mathcal{MD}_{T-1}^T$ and $R_{T-1} = R_{T-1,T}$, it holds

$$\begin{aligned}
&C_{T-1}(X) \\
&= \operatorname*{ess\,inf}_{Q\otimes D \in \mathcal{MD}_{T-1}^T} R_{T-1,T}\Big(X_{[0,T-2]}, Q \otimes D, \\
&\qquad\qquad E_Q\Big[D\Big(X_T - X_{T-1}\Big) + X_{T-1} \,\Big|\, \mathscr{F}_{T-1}\Big]\Big) \\
&= \operatorname*{ess\,inf}_{Q\otimes D \in \mathcal{MD}_{T-1}^T} F_{T-1}(Q \otimes D, X), \qquad (1.35)
\end{aligned}$$

where

$$\begin{aligned}
&F_{T-1}(Q \otimes D) \\
&= \operatorname*{ess\,inf}_{\bar{Q}\otimes \bar{D} \in \mathcal{MD}_T^{t+1}} R_{T-1,T}\Big(X_{[0,T-1]}, Q \otimes D, \\
&\qquad\qquad E_Q\Big[D\Big(F_T(\bar{Q}\otimes\bar{D}, X) - X_{T-1}\Big) + X_{T-1} \,\Big|\, \mathscr{F}_{T-1}\Big]\Big),
\end{aligned}$$

since $F_T(\bar{Q}\otimes\bar{D}, X) = X_T$ for all $\bar{Q}\otimes\bar{D} \in \mathcal{MD}_T^{T+1}$.

For $t = T-2$, by time consistency, and since $C_{T-1}(X)$ is $\mathscr{F}_{T-1}$-measurable, we deduce that

$$\begin{aligned}
&C_{T-2}(X) = C_{T-2}\left(X_{[0,T-2]} + C_{T-1}(X)\,1_{[T-1,T]}\right) \\
&= \operatorname*{ess\,inf}_{Q\otimes D \in \mathcal{MD}_{T-2}^{T-1}} R_{T-2,T-1}\Big(X_{[0,T-3]}, Q \otimes D, \\
&\qquad\qquad E_Q\Big[D\Big(C_{T-1}(X) - X_{T-2}\Big) + X_{T-2} \,\Big|\, \mathscr{F}_{T-2}\Big]\Big).
\end{aligned}$$

Since $s \mapsto R_{T-2,T-1}\left(X_{[0,T-3]}, Q \otimes D, s\right)$ is right-continuous, by means of (1.35) it follows that

$$\begin{aligned}
&R_{T-2,T-1}\Big(X_{[0,T-3]}, Q \otimes D, E_Q\Big[D\Big(C_{T-1}(X) - X_{T-2}\Big) + X_{T-2} \,\Big|\, \mathscr{F}_{T-2}\Big]\Big) \\
&= \operatorname*{ess\,inf}_{\bar{Q}\otimes \bar{D} \in \mathcal{MD}_{T-1}^T} R_{T-2,T-1}\Big(X_{[0,T-3]}, Q \otimes D, \\
&\qquad\qquad E_Q\Big[D\Big(F_{T-1}(\bar{Q}\otimes\bar{D}, X) - X_{T-2}\Big) + X_{T-2} \,\Big|\, \mathscr{F}_{T-2}\Big]\Big) \\
&= F_{T-2}(Q \otimes D, X)
\end{aligned}$$

which ends the proof. □

Remark 1.38. Suppose that α is given by

$$\alpha_t(X) = \sum_{k=1}^{T-1} D'_k \Delta X_k + \beta_t(X_{[t,T]})$$

as in Example 1.32, where $D' = (D_0, \dots, D'_{T-1})$ is fixed, and β is a strongly time consistent path independent assessment index. Then, it follows easily that α itself is a strongly time consistent AI.

1.5 Examples

1.5.1 Dynamic Gain-to-Loss Ratio

We shall discuss here an important example of an assessment index, namely the dynamic Gain-to-Loss Ratio (dGLR). This index, in fact, provides an example of a dynamic acceptability index, since it is scale invariant. It was introduced in [12], in a slightly different form. The version of dGLR given in Definition 1.39 below is not strongly time-consistent in the sense of Definition 1.34, but it is time-consistent in the sense of [12].

The prototype for the definition below is the classical measure of performance Gain-to-Loss Ratio (GLR): given an integrable, real-valued random variable X, GLR is defined as $\mathrm{GLR}(X) := \mathbb{E}(X)/\mathbb{E}(X^-)$ if $\mathbb{E}[X] > 0$, $\mathrm{GLR}(0) = +\infty$ and zero otherwise, where $X^- := \max\{-X, 0\}$. For $\mathrm{GLR} : L^1 \to \overline{\mathbb{R}}$, the set $\mathcal{A}^m = \{X \in L^1 : \mathrm{GLR}(X) \geq m\}$, $m \in \mathbb{R}$, can be expressed as follows. If $m \leq 0$, then $\mathcal{A}^m = L^1$, since $\mathrm{GLR} \geq 0$. If $m > 0$ it holds that $\mathcal{A}^m = \{X \in L^1 : E[X - mX^-] \geq 0\}$. Indeed, $E[X - mX^-] \geq 0$ is equivalent to $E[X] \geq mE[X^-]$ which correspond to $\mathrm{GLR} \geq m$ in the case that $E[X] > 0$, $E[X^-] > 0$. Suppose it holds that $E[X] \geq mE[X^-]$ but $E[X] \leq 0$. Since $mE[X^-] \geq 0$, this yields that $E[X] = 0$. Thus, by $E[X] \geq mE[X^-]$ it follows that $E[X^-] = 0$ as well which can only be the case if $X = 0$. Supposing $E[X^-] = 0$ and $E[X] \geq mE[X^-]$ yields either $E[X] > 0$ or $X = 0$ as before. Since $\mathrm{GLR}(0) = +\infty > m$, the one inclusion is shown. Reversely, if $\mathrm{GLR}(X) \geq m$, then the cases $X = 0$ and $E[X^-] \neq 0$ clearly fulfill $E[X] \geq mE[X^-]$. Supposing $E[X^-] = 0$ however also implies $E[X] \geq mE[X^-]$, as $\mathrm{GLR} \geq m$ demands for $E[X] > 0$ in case that $X \neq 0$. Hence, for $m > 0$, $\mathcal{A}^m$ is the upper level set of the convex, positive homogeneous continuous function $X \mapsto E[X - mX^-]$ and hence a closed, convex cone (the same is true for $\mathcal{A}^m = L^1$, $m \leq 0$).

For the rest of this Section we use the setup of Section 1.3. In particular, we fix $t \in \{1, \dots, T\}$, choose $\mathcal{X} = L^{t,p}(\mathscr{O})$ and consider $\mathcal{X}$ to be an $L^0(\mathscr{O}^t)$-module. Recall that the cone $\mathcal{K}$ in this case is given by $\mathcal{K} = \{X \in \mathcal{X} : X \geq 0\}$. Here, any element

$X \in \mathcal{X}$ is considered to be a discounted dividend process and the next definition provides a relevant formula for dGLR.

Definition 1.39. Let X represent the discounted dividend process. We define dGLR as follows

$$dGLR_s(X) = \begin{cases} G(X), & s \geq t \\ +\infty, & s \leq t-1, \end{cases}$$

where

$$G(X) = \begin{cases} \frac{E\left[\sum_{s=t}^{T} X_s \,|\, \mathscr{F}_t\right]}{E\left[\left(\sum_{s=t}^{T} X_s\right)^{-} \,|\, \mathscr{F}_t\right]}, & \text{on } B_1^X \\ +\infty, & \text{on } B_2^X \\ 0, & \text{on } B_3^X. \end{cases}$$

with $B_1^X := \{E[\sum_{s=t}^{T} X_s \,|\, \mathscr{F}_t] > 0\}$, $B_2^X := \operatorname{ess\,sup}\{A \in \mathscr{F}_t : 1_A \sum_{s=t}^{T} X_s = 0\}$, and $B_3^X := (B_1^X \cup B_2^X)^c$.

Note that for any $X \in \mathcal{X}$, we have that $P(B_1^X \cap B_2^X) = 0$, hence G is well defined. We will show that the above dGLR is monotone, quasi-concave, local, scale invariant and upper semicontinuous. It clearly suffices to show that these properties are satisfied for the function G.

Consider $\mathcal{A}^Y = \{X \in L^{t,p}(\mathscr{O}) : G(X) \geq Y\}$ for $Y \in L^0(\mathscr{O}^t)$. Define $B := \{Y > 0\}$. Since $G(X) \geq 0$, it follows that $1_{B^c} G(X) \geq 1_{B^c} Y$ for every X. Moreover, it holds that

$$\mathcal{A}^Y = \left\{ X : 1_B \left(E[\sum_{s=t}^{T} X_s \,|\, \mathscr{F}_t] - Y E[\left(\sum_{s=t}^{T} X_s\right)^{-} \,|\, \mathscr{F}_t] \right) \geq 0 \right\}.$$

Indeed, we first note that an element X in $\mathcal{A}^Y$ or in the set on the right-hand side above necessarily has to fulfill $B \subseteq B_1^X \cup B_2^X$, as $B_3^X = \left\{ E[\sum_{s=t}^{T} X_s \,|\, \mathscr{F}_t] \leq 0, \sum_{s=t}^{T} X_s \neq 0 \right\}$. On $B_1^X \cup B_2^X$, however, we argue as in the unconditional case to obtain the equality. Hence, $\mathcal{A}^Y$ is the upper level set of a local, (L^0)-quasiconcave, -positively homogeneous function. For a more detailed proof of all these properties compare [10].

We apply Proposition 1.15 to obtain the following result.

Proposition 1.40. *The unique minimal risk function R in representation* (1.6) *of GLR* *has the following form*

$$R(Z,s)=\begin{cases}+\infty, & \text{if } s\geq 0\\ \frac{b_Z}{a_Z}-1, & \text{if } -\infty<s<0,\\ -\infty, & \text{if } s=-\infty.\end{cases}$$

where $a_Z:=\sup\{r\in\mathbb{R}:r\leq Z\}$ *and* $b_Z:=\inf\{r\in\mathbb{R}:Z\leq r\}$.

Proof. Let α be the GLR. Then, from [25], we know that

$$\alpha(X)=\sup\left\{m\geq 0\ :\ \inf_{Q\in\mathcal{Q}^m}E^Q[X]\geq 0\right\},$$

where the system of supporting kernels $\{\mathcal{Q}^m\}_{m\in\mathbb{R}_+}$ for α is given explicitly by (compare [25, Proposition 4])

$$\mathcal{Q}^m=\{c(1+Y):c\in\mathbb{R}_+,\ 0\leq Y\leq m,\ E[c(1+Y)]=1\}\,,\quad m\in\mathbb{R}_+.$$

Using this, it can be verified that

$$\mathcal{A}^{m,\circ}=\{Z\in L^\infty : c\leq Z\leq c(m+1)\text{ for some }c\in\mathbb{R}_+\}\,.$$

Clearly, $\inf\{m\in\bar{\mathbb{R}}:Z\in\mathcal{A}^{m,\circ}\}=b_Z/a_Z-1$, and so, using Proposition 1.15 we conclude the proof. □

Analogously one can establish a robust representation for dGLR.

1.5.2 Optimized Certainty Equivalent

We sketch here a conditional version of classical version of the optimized certainty equivalent. The detailed study can be done along the lines of the study the we conducted above for dGLR.

The optimized certainty equivalent, compare [6, 7], is an assessment index given by

$$OCE_t(X)=\operatorname*{ess\,sup}_{m\in L^\infty(\mathscr{F}_t)}\left\{m+E_{\tilde{P}}\left[u_t\left(X_{[t,T]}-m1_{[t,T]}\right)\ \middle|\ \mathscr{F}_t\right]\right\},$$

where $u_t:\mathbb{R}\to\mathbb{R}\cup\{-\infty\}$ is a concave utility function[22] such that $u(0)=0$ and $1\in\partial u(0)$. Following the same argumentation as in [7, 33], it follows that the robust representation is of the form

$$R\left(Q\otimes D,m\right)=m+E_P\left[\sum_{k=t+1}^{T}\varphi_t\left(\frac{M_k\gamma_k}{M_t\mu_k}\right)\ \middle|\ \mathscr{F}_t\right],\quad Q\otimes D\in\mathcal{M}\otimes_t\mathcal{D},$$

[22] One may assume that u_t can be made $\mathscr{F}_t$-state dependent. This however only a technical step.

where φ_t is the convex conjugate of $-u(-\cdot)$, M is the density process of Q and $Q \otimes \gamma = Q \otimes D$ by means of relation (1.25).

As for the dynamic, of the OCE, if $u_t(x) = (1 - e^{-\gamma x})/\gamma$, for a fixed γ, then the OCE is the entropy and is time consistent, compare [2]. Otherwise, being a risk measure it is a certainty equivalent, henceforth, a recursive definition along the line of Proposition 1.36 yields a strong time consistent assessment index.

1.5.3 Weighted $V@R$ Acceptability Indices

Similarly as in the previous subsection we present here just a sketch of possible conditional version of weighted $V@R$ acceptability indices .

Following [25], we define $[0,1](\mathscr{F}_t) = \{\alpha \in L^0(\mathscr{F}_t) : 0 \leq \alpha \leq 1\}$. This set is clearly σ-stable. We consider a family of functions $\Phi_m : [0,1](\mathscr{F}_t) \to [0,1](\mathscr{F}_t)$, $m \in L^0_+(\mathscr{F}_t)$ being

- jointly local:
$$1_A \Phi_m(\alpha) + 1_{A^c}\Phi_n(\beta) = \Phi_{1_A m + 1_{A^c} n}(1_A \alpha + 1_{A^c}\beta),$$
for every $A \in \mathscr{F}_t$, $m, n \in L^0_+(\mathscr{F}_t)$ and $\alpha, \beta \in [0,1](\mathscr{F}_t)$;
- concave: $\alpha \mapsto \Phi_m(\alpha)$ is concave;
- increasing: $\Phi_m \leq \Phi_n$, for every $m \leq n \in L^0_+(\mathscr{F}_t)$;
- normalized: $\Phi_m(0) = 0$ and $\Phi_m(1) = 1$, for every $m \in L^0_+(\mathscr{F}_t)$.

Such a family is called a conditional family of concave distortions. Note that being conditionally concave and local, it follows that Φ_m is continuous. We define the Weighted $V@R$ acceptability index as follows

$$AIW(X) := \operatorname{ess\,sup}\left\{ m \in L^0_+(\mathscr{F}_t) : \int_{-\infty}^{\infty} x d\Psi_m\left(F_{(X_{[t+1,T]}\,|\,\mathscr{F}_t)}(x)\right) \geq 0 \right\},$$

where $F_{(X_{[t+1,T]}\,|\,\mathscr{F}_t)}(x) = \tilde{P}[X_{[t+1,T]} \leq x \,|\, \mathscr{F}_t]$ is the regular conditional distribution under $\tilde{P}$ of $X_{[t+1,T]}$, the integral being taken ω-wise. Once again, following the argumentation in [25], it follows that

$$\begin{aligned}\mathcal{A}_1^{m,\circ} : = \Big\{ Q \otimes D \in \mathcal{M} \otimes_t \mathcal{D} : E\left[\left(-\sum_{k=t+1}^{T} M_k \Delta D_{k+1} - \mu_k \beta\right) \,\Big|\, \mathscr{F}_t\right] \leq \phi_m(\beta), \\ \text{for all } \beta \in L^0_+(\mathscr{F}_t)\Big\},\end{aligned}$$

where M is the density process of Q, $\phi_m(\beta) := \operatorname{ess\,sup}_{\alpha \in [0,1](\mathscr{F}_t)}\{\Phi_m(\alpha) - \alpha\beta\}$, $m \in L^0_+(\mathscr{F}_t)$ and $\beta \in L^0_+(\mathscr{F}_t)$ is the convex conjugate of Φ_m.[23] With this formulation, one may define $AIMAX$, $AIMAXMIN$, $AIMINMAX$.

1.6 Appendix

1.6.1 Standard Results on L^0-Convex Analysis

Notations and settings are from the Preliminaries 1.1. Let $\mathcal{Y}$ be a set of L^0-linear functionals from $\mathcal{X}$ to L^0. We denote by L^0-$\sigma(\mathcal{X},\mathcal{Y})$ the smallest topology for which the mappings

$$X \mapsto Z(X), \quad X \in \mathcal{X}$$

are L^0-continuous for any $Z \in \mathcal{Y}$.

Proposition 1.41. *Let $\mathcal{X}$ be a locally L^0-convex topological L^0-module and let $\mathcal{Y}$ be a set of L^0-linear functionals from $\mathcal{X}$ to L^0. Then, $\mathcal{X}$ equipped with the L^0-$\sigma(\mathcal{X},\mathcal{Y})$-topology is a locally L^0-convex topological L^0-module.*

Proof. By definition, the L^0-$\sigma(\mathcal{X},\mathcal{Y})$-topology on $\mathcal{X}$ is generated by the following family of neighborhoods of 0

$$U_{\mathcal{A},\varepsilon} := \left\{X \in \mathcal{X} : \sup_{Z \in \mathcal{A}} |Z(X)| \le \varepsilon\right\},$$

where $\mathcal{A}$ is a finite subset of $\mathcal{Y}$ and $\varepsilon \in L^0_{++}$. Since $|Z(.)|$ is an L^0-seminorm[24], we apply [43, Theorem 2.4]. □

Provided that $\mathcal{Y}$ is itself an L^0-module, $\mathcal{X}$ also defines a set of L^0-linear functionals from $\mathcal{Y}$ to L^0 and therefore $\left(\mathcal{Y}, L^0\text{-}\sigma(\mathcal{Y},\mathcal{X})\right)$ is again a locally L^0-convex topological L^0-module. Furthermore, the L^0-dual space of $\left(\mathcal{X}, L^0\text{-}\sigma(\mathcal{X},\mathcal{Y})\right)$ is exactly $\mathcal{Y}$. We finally say that $\mathcal{X}$ is L^0-reflexive if $\mathcal{X}^{**} = \mathcal{X}$, in which $\mathcal{X}^*$ is equipped with the L^0-$\sigma(\mathcal{X}^*,\mathcal{X})$-topology. On $\mathcal{X}^* \times \mathcal{X}$ we always consider the dual pairing $\langle X^*, X\rangle := X^*(X)$.

A local function $F : \mathcal{X} \to \bar{L}^0$ is said to be

- upper semicontinuous if the upper level sets given by $\{X \in \mathcal{X} : F(X) \ge m\}$ are closed for all $m \in \bar{L}^0$;
- proper if $F < \infty$ and there exists $X \in \mathcal{X}$ such that $F(X) > -\infty$.

[23] Clearly, (ϕ_m) is a jointly local family of convex increasing functions.
[24] An L^0-seminorm is a functional $p : \mathcal{E} \to L^0_+$ such that $p(mX) = |m|\,p(X)$ for any $m \in L^0$ and $X \in \mathcal{E}$ and $p(X+Y) \le p(X) + p(Y)$ for any $X, Y \in \mathcal{E}$.

The concave conjugate $F^{\star} : \mathcal{X}^* \to \bar{L}^0$ of F is given by

$$F^{\star}(X^*) := \operatorname*{ess\,inf}_{X \in \mathcal{X}} \{\langle X^*, X\rangle - F(X)\}, \quad X^* \in \mathcal{X}^*.$$

The hypograph hypo (F) of F is defined as

$$\text{hypo}\,(F) := \left\{(X, m) \in \mathcal{X} \times L^0 : F(X) \geq m\right\}.$$

From now on we consider $\mathcal{X}$ to be a σ-stable, locally L^0-convex topological L^0-module such that the set of all neighborhoods of zero is σ-stable. From the theory of L^0-modules in [43] we know the following.

Proposition 1.42. *Let $F : \mathcal{X} \to \bar{L}^0$ be a proper function, then*

1. *F is L^0-concave if and only if* hypo (F) *is L^0-convex and F is L^0-local.*
2. *$F^{\star}$ is L^0-concave and L^0-upper semicontinuous for any F.*
3. *If F is an L^0-proper concave upper semicontinuous function then $F^{\star\star} = F$.*

Definition 1.43. For a nonempty family $(A_i)_{i \in I} \in \mathscr{G}$ the essential supremum, denoted by $\operatorname{ess\,sup}\{A_i : i \in I\}$, is defined to be the element $B \in \mathscr{G}$ with

1. $A_i \subseteq B$ for all i.
2. For all $C \in \mathscr{G}$ fulfilling 1. and $C \subseteq B$ holds $P[B \setminus C] = 0$.

Further we define $\operatorname{ess\,sup}\{(\emptyset)\} = \emptyset$.

The next lemma was proven in [43, Lemma 2.9].

Lemma 1.44. *Every nonempty family $\mathscr{A} = (A_i)_{i \in I}$ has an essential supremum. If for all i, j also $A_i \cup A_j \in \mathscr{A}$, then there exists an increasing sequence (A^n) in $\mathscr{A}$ such that* $\operatorname{ess\,sup}(\mathscr{A}) = \bigcup_{n \in \mathbb{N}} A^n$.

1.6.2 Conditional Inverse of Increasing Functions

In this section, for $n, m \in \bar{L}^0$, we use the convention that $n < m$ if $P[n < m] = 1$.

For a local, increasing[25] function $F : \bar{L}^0 \to \bar{L}^0$ we define its left- and right-continuous version as

$$F^-(m) := 1_{\widetilde{A}_m} \operatorname{ess\,sup}\left\{F(n) : n \in \bar{L}^0 \text{ and } n < m \text{ on } \widetilde{A}_m\right\} - 1_{\widetilde{A}_m^c}\infty, \tag{1.36}$$

$$F^+(m) := 1_{\widetilde{B}_m} \operatorname{ess\,inf}\left\{F(n) : n \in \bar{L}^0 \text{ and } n > m \text{ on } \widetilde{B}_m\right\} + 1_{\widetilde{B}_m^c}\infty, \tag{1.37}$$

[25] That is, $F(m) \geq F(m')$ whenever $m \geq m'$.

where $m \in \bar{L}^0$ and $\widetilde{A}_m = \{m > -\infty\}$, and $\widetilde{B}_m = \{m < \infty\}$. Due to locality and the definition of $F^{\pm}$ it holds that

$$F^{+}(m) \leq F^{-}(m'), \quad \text{for } m, m' \in \bar{L}^0 \text{ with } m < m'. \tag{1.38}$$

Definition 1.45. For a local, increasing function $F : \bar{L}^0 \to \bar{L}^0$, a local, increasing function $G : \bar{L}^0 \to \bar{L}^0$ is called a conditional inverse of F if

$$\begin{cases} F^{-}\left(G(s)\right) \leq s \leq F^{+}\left(G\left(s\right)\right) & \text{, on } \left\{F(-\infty) < s < F(\infty)\right\}, \\ G(s) = -\infty & \text{, on } \left\{s < F(-\infty)\right\}, \\ G(s) = \infty & \text{, on } \left\{F(\infty) < s\right\}, \end{cases} \tag{1.39}$$

for every $s \in \bar{L}^0$.

Remark 1.46. The definition of a conditional inverse does not postulate any condition as for the values of G on the boundary of the range of F. Being increasing, it simply means that $-\infty \leq G(F(-\infty)) \leq G^{+}(F(-\infty))$ and $G^{-}(F(\infty)) \leq G(F(\infty)) \leq +\infty$. We can not require $G(F(-\infty)) = -\infty$ or $G(F(-\infty)) = G^{+}(F(-\infty))$ for instance. This is important since by the definition of the left- and right-inverse below, Proposition 1.49 states that both $F^{(-1,l)}$ and $F^{(-1,r)}$ are inverses of F. However, it may well happen that $F^{(-1,l)}(F(-\infty)) = -\infty < F^{(-1,r)}(F(-\infty))$ as well as $F^{(-1,l)}(F(\infty)) < F^{(-1,r)}(F(\infty)) = +\infty$ and a convention on the values of a conditional inverse on the boundaries of F would imply that neither $F^{(-1,l)}$ and $F^{(-1,r)}$ are conditional inverse.

We define the conditional left- and right-inverse of F as

$$\begin{aligned} F^{(-1,l)}\left(s\right) :=& 1_{A_s} \operatorname{ess\,inf}\{m \in \bar{L}^0 : 1_{A_s} F\left(m\right) \geq 1_{A_s} s\} + 1_{A_s^c}\infty \\ =& 1_{A_s} \operatorname{ess\,sup}\{m \in \bar{L}^0 : F\left(m\right) < s \text{ on } A_s\} + 1_{A_s^c}\infty, \\ F^{(-1,r)}\left(s\right) :=& 1_{B_s} \operatorname{ess\,sup}\{m \in \bar{L}^0 : 1_{B_s} F\left(m\right) \leq 1_{B_s} s\} - 1_{B_s^c}\infty \\ =& 1_{B_s} \operatorname{ess\,inf}\{m \in \bar{L}^0 : F\left(m\right) > s \text{ on } B_s\} - 1_{B_s^c}\infty, \end{aligned}$$

for $s \in \bar{L}^0$, where[26] $A_s := \{F(\infty) \geq s\}$ and $B_s := \{F(-\infty) \leq s\}$.

Lemma 1.47. *The conditional left- and right-inverse of a local, increasing function $F : \bar{L}^0 \to \bar{L}^0$ are local, increasing functions which are left- and right-continuous, respectively.*

Proof. Consider a local, increasing function $F : \bar{L}^0 \to \bar{L}^0$. We will prove the statement for the left-inverse $F^{(-1,l)}$, and the case of right-inverse function is done similarly.

Step 1: Note that $A_{\tilde{s}} \supseteq A_s$ for every $\tilde{s} \leq s$. This implies that $1_{A_s^c}\infty$ is increasing. Hence, a direct inspection shows that $F^{(-1,l)}$ is increasing.

[26]Note that $A_s := \left\{\operatorname{ess\,sup}_{m \in \bar{L}^0} F(m) < s\right\}^c$ and $B_s := \left\{\operatorname{ess\,inf}_{m \in \bar{L}^0} F(m) > s\right\}^c$

Step 2: Next we will show that $F^{(-1,l)}$ is local. Pick $s, \tilde{s} \in \bar{L}^0$ and $B \in \mathscr{G}$. Since F is local, it follows that

$$C^c := A^c_{1_B s + 1_{B^c}\tilde{s}} = \{F(\infty) < 1_B s + 1_{B^c}\tilde{s}\} = (B \cap A^c_s) \cup (B^c \cap A^c_{\tilde{s}}).$$

Consequently, we deduce that $C = (B \cap A_s) \cup (B^c \cap A_{\tilde{s}}) \cup (A_s \cap A_{\tilde{s}})$. However, $(A_s \cap A_{\tilde{s}}) \subseteq (B \cap A_s) \cup (B^c \cap A_{\tilde{s}})$, hence

$$C = (B \cap A_s) \cup (B^c \cap A_{\tilde{s}}). \tag{1.40}$$

This implies that

$$1_C(1_B s + 1_{B^c}\tilde{s}) = 1_B 1_{A_s} s + 1_{B^c} 1_{A_{\tilde{s}}} \tilde{s} \tag{1.41}$$
$$1_{C^c}(1_B s + 1_{B^c}\tilde{s}) = 1_B 1_{A^c_s} s + 1_{B^c} 1_{A^c_{\tilde{s}}} \tilde{s}. \tag{1.42}$$

We claim that,

$$1_B \left\{ m \in \bar{L}^0 : 1_B 1_{A_s} F(m) \geq 1_B 1_{A_s} s \right\} = 1_B \left\{ m \in \bar{L}^0 : 1_{A_s} F(m) \geq 1_{A_s} s \right\}. \tag{1.43}$$

Indeed, inclusion $\supseteq$ is straightforward. For the converse inclusion, let $1_B \tilde{n}$ be in $1_B \left\{ m \in \bar{L}^0 : 1_B 1_{A_s} F(m) \geq 1_B 1_{A_s} s \right\}$. Note that by the definition of A_s, the set $\{m \in \bar{L}^0 : 1_{A_s} F(m) \geq 1_{A_s} s\}$ is not empty. Indeed, $A_s = \{F(\infty) \geq s\}$, hence, $1_{A_s} F(\infty) \geq 1_{A_s} s$ showing that $\infty \in \{m \in \bar{L}^0 : 1_{A_s} F(m) \geq 1_{A_s} s\}$. Hence, pick some $\tilde{m}$ in $\left\{ m \in \bar{L}^0 : 1_{A_s} F(m) \geq 1_{A_s} s \right\}$. Locality of F yields that $1_B \tilde{n} + 1_{B^c} \tilde{m}$ is in the set $\left\{ m \in \bar{L}^0 : 1_{A_s} F(m) \geq 1_{A_s} s \right\}$. Multiplying by 1_B, we obtain that also $1_B \tilde{m}$ is in the set $1_B \left\{ m \in \bar{L}^0 : 1_{A_s} F(m) \geq 1_{A_s} s \right\}$.

Using (1.40)-(1.43), and locality of F, we deduce

$$\begin{aligned}
F^{(-1,l)}(1_B s + 1_{B^c}\tilde{s}) =& 1_C \operatorname{ess\,inf} \left\{ m \in \bar{L}^0 : 1_C F(m) \geq 1_C (1_B s + 1_{B^c}\tilde{s}) \right\} + 1_{C^c}\infty \\
=& 1_B 1_{A_s} \operatorname{ess\,inf} \left\{ m \in \bar{L}^0 : 1_B 1_{A_s} F(m) \geq 1_B 1_{A_s} s \right\} \\
&+ 1_{B^c} 1_{A_{\tilde{s}}} \operatorname{ess\,inf} \left\{ m \in \bar{L}^0 : 1_{B^c} 1_{A_{\tilde{s}}} F(m) \geq 1_{B^c} 1_{A_{\tilde{s}}} \tilde{s} \right\} \\
&+ 1_B 1_{A^c_s}\infty + 1_{B^c} 1_{A^c_{\tilde{s}}}\infty \\
=& 1_B \left(1_{A_s} \operatorname{ess\,inf} \left\{ m \in \bar{L}^0 : 1_{A_s} F(m) \geq 1_{A_s} s \right\} + 1_{A^c_s}\infty \right) \\
&+ 1_{B^c} \left(1_{A_{\tilde{s}}} \operatorname{ess\,inf} \left\{ m \in \bar{L}^0 : 1_{A_{\tilde{s}}} F(m) \geq 1_{A_{\tilde{s}}} \tilde{s} \right\} + 1_{A^c_{\tilde{s}}}\infty \right) \\
=& 1_B F^{(-1,l)}(s) + 1_B F^{(-1,l)}(\tilde{s}).
\end{aligned}$$

Hence $F^{(-1,l)}$ is local.

Step 3: Finally, we will show that $F^{(-1,l)}$ is left-continuous. Let $s \in \bar{L}^0$.

By the definition of $F^{(-1,l)}$ and locality of F, clearly $F^{(-1,l)}(s) = -\infty$ on the set

$C_s^c = \{s = -\infty\}$. Consider now $D_s = C_s \cap \{F(\infty) \geq s\} = \{s > -\infty\} \cap \{F(\infty) \geq s\}$, and $D_{\tilde{s}} := C_s \cap \{F(\infty) \geq \tilde{s}\}$, for some $\tilde{s} \in \bar{L}^0$. Note that $D_s \subseteq D_{\tilde{s}}$ for any $\tilde{s}$ such that $\tilde{s} < s$ on D_s.

Denote by $\mathcal{S}$ the set of those $\tilde{s} \leq s$ such that $\tilde{s} < s$ on D_s. Note that $\mathcal{S} \neq \emptyset$. Let $\tilde{s} \in \mathcal{S}$, and suppose that $\operatorname{ess\,sup}_{\tilde{s}\in\mathcal{S}} F^{(-1,l)}(\tilde{s}) < \tilde{m} < F^{(-1,l)}(s)$ on some set $D \subseteq D_s$. By the definition of the left-inverse, and locality of F, it follows that $\tilde{s} < F(\tilde{m}) < s$ on D for every $\tilde{s} \in \mathcal{S}$, which is not possible unless $P[D] = 0$. Hence, $\operatorname{ess\,sup}_{\tilde{s}\in\mathcal{S}} F^{(-1,l)}(\tilde{s}) = F^{(-1,l)}(s)$ on D_s. From here, using locality of F, we also have that $\operatorname{ess\,sup}_{\tilde{s}<s} F^{(-1,l)}(\tilde{s}) = F^{(-1,l)}(s)$ on D_s. Next, let us consider the set $E_s := C_s \cap \{F(\infty) < s\}$. Since $F(\infty) < s$ on E_s, there exists $\tilde{s} \in \bar{L}^0$ such that $\tilde{s} < s$ on E_s, and $E_{\tilde{s}} := C_s \cap \{F(\infty) < \tilde{s}\} = E_s$. Therefore, by the definition of $F^{(-1,l)}$ we conclude that $F^{(-1,l)}(\tilde{s}) = F^{(-1,l)}(s)$ for any $\tilde{s} < s$ on $E_s = E_{\tilde{s}}$, which consequently shows that $F^{(-1,l)}$ is left continuous on E_s.
Finally, since C_s^c, D_s, E_s forms a partition of Ω, and $F^{(-1,l)}$ is left-continuous on each of the sets from the partition, combined with locality of $F^{(1-,l)}$, we deduce that $F^{(-1,l)}$ is left-continuous.

The case of $F^{(-1,r)}$ follows analogously. □

Remark 1.48. The sets A_s, B_s are used to guarantee the locality of the right-and left-inverse, respectively. Indeed, suppose that we would define $F^{(-1,l)}(s) = \operatorname{ess\,inf}\{m \in \bar{L}^0 : F(m) \geq s\}$. Then it is possible to get a nonlocal inverse. For example, let $A \in \mathscr{G}$ with $0 < P[A] < 1$ and $F(m) := 1_A 2m + 1_{A^c}$ which is increasing and local. Then, $F^{(-1,l)}(1_A 2) = 1_A - 1_{A^c}\infty$, whereas $F^{(-1,l)}(2) = \operatorname{ess\,inf}\emptyset = +\infty$, and thus $1_A F^{(-1,l)}(1_A 2) = 1_A \neq 1_A\infty = 1_A F^{(-1,l)}(2)$, which implies that $F^{(-1,l)}$ would not be local.

Proposition 1.49. *Let $F : \bar{L}^0 \to \bar{L}^0$ be a local, increasing function. Then, the following properties hold true:*

(i) Any conditional inverse G of F satisfies

$$F^{(-1,l)} = G^- \leq G \leq G^+ = F^{(-1,r)}; \tag{1.44}$$

(ii) $F^{(-1,l)}$ and $F^{(-1,r)}$ are also both conditional inverse of F;

(iii) F is a conditional inverse of any of its conditional inverses;

(iv) For any $m, s \in \bar{L}^0$ we have that

$$F^-(m) \leq s \iff m \leq F^{(-1,r)}(s) \tag{1.45}$$

$$F^+(m) \geq s \iff m \geq F^{(-1,l)}(s). \tag{1.46}$$

Remark 1.50. Note that since F is a conditional inverse of any of its conditional inverse, (1.44) implies that

$$F^- = \left(F^{(-1,l)}\right)^{(-1,l)} = G^{(-1,l)} = \left(F^{(-1,r)}\right)^{(-1,l)},$$
$$F^+ = \left(F^{(-1,r)}\right)^{(-1,r)} = G^{(-1,r)} = \left(F^{(-1,l)}\right)^{(-1,r)}.$$

Proof. Consider a local, increasing function $F : \bar{L}^0 \to \bar{L}^0$ and a conditional inverse G of F.

Step 1: Let us show that

$$F^{(-1,l)} \leq G^- \leq G \leq G^+ \leq F^{(-1,r)}. \tag{1.47}$$

The fact that $G^- \leq G \leq G^+$ follows from the definition of left- and right-continuous version and from the fact that G is increasing. By Lemma 1.47, we have that $F^{(-1,l)}$ and $F^{(-1,r)}$ are local, increasing and left- and right-continuous, respectively.

Let us show now that $F^{(-1,l)} \leq G^-$. Since $F^{(-1,l)}$ is left-continuous, and both $F^{(-1,l)}$ and G are increasing, it is sufficient to show that $F^{(-1,l)}(s) \leq G(s)$, for every $s \in \bar{L}^0$. Assume that $s \in \bar{L}^0$. The definition of $F^{(-1,l)}$ shows that $F^{(-1,l)}(s) = -\infty \leq G(s)$ on $\{s \leq F(-\infty)\}$. Since G is an inverse of F, it follows that $G(s) = \infty \geq F^{(-1,l)}(s)$ on[27] $\{s > F(\infty)\}$. On $\{F(-\infty) < s < F(\infty)\}$, suppose that there exists $\tilde{m} \in L^0$ such that $F^{(-1,l)}(s) > \tilde{m} > G(s)$ on some set $A \subseteq \{F(-\infty) < s < F(\infty)\}$. On the one hand, by definition of $F^{(-1,l)}$ follows that $s > F(\tilde{m})$ on A. On the other hand, since $\tilde{m} > G(s)$ on A it follows by means of (1.38) that $F(\tilde{m}) \geq F^-(\tilde{m}) \geq F^+(G(s))$. Thus, $s > F^+(G(s))$ on $A \subseteq \{F(-\infty) < s < F(\infty)\}$, which contradicts the fact that G is an inverse of F. Hence, A has to be of probability 0, and so, we proved that $F^{(-1,l)} \leq G$ on $\{F(-\infty) < s < F(\infty)\}$.

Finally, note that since $F^{(-1,l)}$ is left-continuous and $F^{(-1,l)}(s') \leq G(F(\infty))$ for any $s' < F(\infty)$, we have that $F^{(-1,l)}(F(\infty)) \leq G(F(\infty))$. The latter, together with locality of $F^{(-1,l)}$ and G, imply that $F^{(-1,l)}(s) \leq G(s)$ on set $\{s = F(\infty)\}$. Hence, we conclude that $F^{(-1,l)} \leq G$.

A similar argumentation shows that $G^+ \leq F^{(-1,r)}$ and therefore (1.47) holds true.

Step 2: Let us show that

$$(F^{(-1,l)})^+ = F^{(-1,r)} \tag{1.48}$$
$$(F^{(-1,r)})^- = F^{(-1,l)}. \tag{1.49}$$

[27] Here also the set to be considered is $\{s > F(\infty)\}$ and not $\{s \geq F(\infty)\}$.

Since $F^{(-1,l)} \leq F^{(-1,r)}$ and the latter is right-continuous, it follows that $(F^{(-1,l)})^+ \leq F^{(-1,r)}$. Reversely, for any $s < \tilde{s}$, we have that $F^{(-1,r)}(s) \leq F^{(-1,l)}(\tilde{s})$. Indeed, on $A^c_{\tilde{s}}$, it holds $F^{(-1,l)}(\tilde{s}) = \infty \geq F^{(-1,r)}(s)$. On B^c_s, it holds $F^{(-1,r)}(\tilde{s}) = -\infty \leq F^{(-1,l)}(s)$. Finally, on $C = (A^c_{\tilde{s}} \cup B^c_s)^c = A_{\tilde{s}} \cap B_s$, it holds $F(-\infty) \leq s < \tilde{s} \leq F(\infty)$. Using now $s < \tilde{s}$, and the definition of $F^{(-1,l)}$ and $F^{(-1,r)}$, since $C \subseteq A_{\tilde{s}}$ and $C \subseteq B_s$, it yields

$$1_C \left\{ m \in \bar{L}^0 : F(m) > s \text{ on } C \right\} \supseteq 1_C \left\{ m \in \bar{L}^0 : 1_C F(m) \geq 1_C \tilde{s} \right\}.$$

Taking the essential infimum on both sides shows that $1_C F^{(-1,r)}(s) \leq 1_C F^{(-1,l)}(\tilde{s})$ for any $s < \tilde{s}$. This together with $(F^{(-1,l)})^+ \leq F^{(-1,r)}$ implies by the definition of the right-continuous version that $1_C (F^{(-1,l)})^+ = 1_C F^{(-1,r)}$. Since $P[C \cup A^c_{\tilde{s}} \cup B^c_s] = 1$, it follows that $(F^{(-1,l)})^+ = F^{(-1,r)}$.

A similar argumentation yields $(F^{(-1,r)})^- = F^{(-1,l)}$.

Step 3: We deduce from (1.47), (1.48) and (1.49) that $F^{(-1,l)} = G^-$ and $F^{(-1,r)} = G^+$. Therefore, (1.44) follows.

Let us prove that $F^{(-1,l)}$ and $F^{(-1,r)}$ are both conditional inverses of F. Towards this end, we first observe that (1.47) together with Lemma 1.47 yield that G^- and G^+ are local, increasing functions. Since $G(s) = -\infty$ on $\{s < F(-\infty)\}$ and $G(s) = \infty$ on $\{s > F(\infty)\}$, it follows immediately that the same holds for the left- and right-continuous versions of G. Using the fact that G is a conditional inverse, monotonicity of $F^{\pm}$ yields

$$F^-(G^-(s)) \leq F^-(G(s)) \leq s \leq F^+(G(s)) \leq F^+(G^+(s)),$$

on $\{F(-\infty) < s < F(\infty)\}$.

Reversely, since F^-, F^+, G are increasing, and G is a conditional inverse, we deduce that

$$\begin{aligned} F^-(G^+(s)) &= F^-(\operatorname*{ess\,inf}_{\tilde{s}>s} G(\tilde{s})) \leq \operatorname*{ess\,inf}_{\tilde{s}>s} F^-(G(\tilde{s})) \leq \operatorname*{ess\,inf}_{\tilde{s}>s} \tilde{s} = s; \\ F^+(G^-(s)) &= F^+(\operatorname*{ess\,sup}_{\tilde{s}<s} G(\tilde{s})) \geq \operatorname*{ess\,sup}_{\tilde{s}<s} F^+(G(\tilde{s})) \geq \operatorname*{ess\,sup}_{\tilde{s}<s} \tilde{s} = s, \end{aligned}$$

on $\{F(-\infty) < s < F(\infty)\}$. Thus $F^{(-1,l)} = G^-$ and $F^{(-1,r)} = G^+$ are both conditional inverse of F.

Step 4: Let us show that F is a conditional inverse of any of its conditional inverses. Let G be a conditional inverse of F and let $s, m \in \bar{L}^0$.

First, we claim that

$$s > F(m) \text{ implies that } G(s) \geq m. \tag{1.50}$$

Indeed, on $\{s > F(\infty)\}$, $G(s) = \infty \geq m$. Next, note that the assumption $s > F(m)$

implies that $\{s < F(\infty)\} = \{F(-\infty) < s < F(\infty)\}$. Hence, on $\{s < F(\infty)\}$, we have that $F^+(G(s)) \geq s > F(m)$ which implies $G(s) \geq m$. Finally, on $\{s = F(\infty)\}$, it follows that $F(\infty) = s > F(m)$. Therefore, $G(F(\infty)) \geq F^{(-1,l)}(F(\infty)) = \operatorname{ess\,inf}\{n \in \bar{L}^0 : F(n) \ \geq F(\infty)\} \geq m$, since $F(\infty) > F(m)$. Hence, (1.50).

By (1.50), and the definition of the right-continuous version, it follows that $G^+(F(m)) \geq m$. A similar argumentation shows that $G^-(F(m)) \leq m$. Consequently,

$$G^-(F(m)) \leq m \leq G^+(F(m)) \quad \text{on } \{G(-\infty) < m < G(\infty)\}. \tag{1.51}$$

Clearly $G(\infty) < \infty$ on set $\{m > G(\infty)\}$. By the definition of the conditional inverse, it follows that $F(\infty) = \infty$, on $\{m > G(\infty)\}$. Hence, by the first line of (1.39) we conclude that $F^+(G(\infty)) = \infty$ on $\{m > G(\infty)\}$, which consequently implies that $F(m) = \infty$ on $\{m > G(\infty)\}$. Similarly, we get that $F(m) = -\infty$ on set $\{m < G(-\infty)\}$. From here, and (1.51), we conclude that F is a conditional inverse of any of its conditional inverse.

Step 5: Finally, let us show that (1.45) and (1.46) are satisfied.

Consider $m, s \in \bar{L}^0$. By the definition of $F^{(-1,l)}$, we have at once [28] that on the set $\{m > -\infty\}$, $F^-(m) \leq s$ implies $m \leq F^{(-1,r)}(s)$. Clearly, this implication also holds true on the set $\{m = -\infty\}$. Similarly, we deduce that $F^+(m) \geq s$ implies $m \geq F^{(-1,l)}(s)$.

The converse implications follow by applying the last two implications to G and then using (1.44) along with Remark 1.50.

□

1.6.3 Proof of Proposition 1.10

Proof. Let us first observe that Proposition 1.49 implies that there is a one-to-one correspondence between functions $F : \bar{L}^0 \to \bar{L}^0$, that are local, increasing and right-continuous, and their conditional right-inverses. In other words, the conditional right-inverse operator is a bijection between the sets of such functions. From this we deduce that if $\pi : \mathcal{K}^\circ \times \bar{L}^0 \to \bar{L}^0$ is local in the second argument and if it satisfies (b), then, its conditional right-inverse, say $R : \mathcal{K}^\circ \times \bar{L}^0 \to \bar{L}^0$ is local in the second argument and satisfies (ii); moreover, the conditional right-inverse of R is equal to π.

In the rest of the proof we shall show that, additional properties of π are satisfied if and only if corresponding additional properties of R are satisfied, for instance (a)–(b) $\Leftrightarrow$ (i)–(ii), (a)–(b), (c) $\Leftrightarrow$ (i)– (ii), (iii), etc.

We start with showing that R is jointly local if π is jointly local. Consider $X^* \in \mathcal{X}^*$, $s \in \bar{L}^0$ and $A \in \mathscr{G}$. By similar argumentations as in the proof of locality from

[28] Note that by definition of the left-continuous version, $F(-\infty) \leq F^-(m) \leq s$ on $\{m > -\infty\}$

Proposition 1.49, and by the joint locality of π, we deduce that

$$\begin{aligned}
1_A R\left(X^*, s\right) &= 1_A R\left(X^*, 1_A s\right) \\
&= 1_A 1_{B_{1_A s}} \operatorname{ess\,sup}\left\{m \in \bar{L}^0 : 1_{B_{1_A s}} \pi\left(X^*, m\right) \leq 1_{B_{1_A s}} 1_A s\right\} - 1_A 1_{B^c_{1_A s}} \infty \\
&= 1_A 1_{B_{1_A s}} \operatorname{ess\,sup}\left\{m \in \bar{L}^0 : 1_A 1_{B_{1_A s}} \pi\left(X^*, m\right) \leq 1_{B_{1_A s}} 1_A s\right\} - 1_A 1_{B^c_{1_A s}} \infty \\
&= 1_A 1_{B_{1_A s}} \operatorname{ess\,sup}\left\{m \in \bar{L}^0 : 1_{B_{1_A s}} \pi\left(1_A X^*, m\right) \leq 1_{B_{1_A s}} 1_A s\right\} - 1_A 1_{B^c_{1_A s}} \infty \\
&= 1_A R\left(1_A X^*, 1_A s\right),
\end{aligned}$$

where $B_{1_A s} = \{\pi(X^*, m) \leq 1_A s\}$. This shows the joint locality of R. Assuming that R is jointly local, the joint locality of π is proved similarly.

The equivalences between (c)–(e) and (iii)–(v) are proved similarly as in [30, Lemma C.2] after corresponding adjustments to the conditional case. Indeed, under condition (a), the fact that $\pi(\cdot, m)$ is upper semicontinuous and concave for every $m \in \bar{L}^0$ is equivalent to the fact that the hypograph of π

$$\left\{(X^*, s) \in \mathcal{K}^\circ \times L^0 : \pi(X^*, m) \geq s\right\}$$

is closed and convex for every $m \in \bar{L}^0$. Using (1.46), this is equivalent to the fact that the set

$$\left\{(X^*, s) \in \mathcal{K}^\circ \times L^0 : m \geq R^-(X^*, s)\right\}$$

is closed and convex for every $m \in \bar{L}^0$, which implies that R^- is jointly lower semicontinuous and quasiconvex. Furthermore R^- is jointly quasiconvex if and only if R is jointly quasiconvex.

Similarly, one can show that π is positive homogeneous if and only if $R(\lambda X^*, s) = R(X^*, s/\lambda)$ for every $\lambda \in L^0_{++}$.

Finally, we will show the equivalence between (d) and (iv), under the assumption that (a), (b), and respectively (i), (ii) are satisfied. Note that condition (d) is equivalent to the following condition

$$\begin{aligned}
&\left(\pi\left(X^*, m\right) = \infty \quad \text{for some } m \in \bar{L}^0,\ X^* \in \mathcal{K}^\circ\right) \\
&\qquad \Longrightarrow \left(\pi\left(Y^*, m\right) = \infty \quad \text{for all } Y^* \in \mathcal{K}^\circ\right),
\end{aligned}$$

which, consequently, is equivalent to

$$\begin{aligned}
&\left(\pi\left(X^*, m\right) \geq s \quad \text{for all } s \in L^0, \text{ and for some } m \in \bar{L}^0,\ X^* \in \mathcal{K}^\circ\right) \\
&\qquad \Longrightarrow \left(\pi\left(Y^*, m\right) \geq s \quad \text{for all } s \in L^0, \text{ and for all } Y^* \in \mathcal{K}^\circ\right).
\end{aligned}$$

By (1.46), it follows that the latter implication is equivalent to

$$\begin{aligned}&\Big(m \geq R^{-}\left(X^{*}, s\right) \quad \text{for all } s \in L^{0}, \text{ for some } m \in \bar{L}^{0},\ X^{*} \in \mathcal{K}^{\circ}\Big)\\ &\quad\Longrightarrow \Big(m \geq R^{-}\left(Y^{*}, s\right) \quad \text{ for all } s \in L^{0},\ \text{and for all } Y^{*} \in \mathcal{K}^{\circ}\Big).\end{aligned}$$

Noticing that $R^{-}(X^{*}, \infty) = \operatorname{ess\,sup}_{s \in L^{0}} R(X^{*}, s)$, we deduce that the last condition is equivalent to

$$\begin{aligned}&\Big(m \geq R^{-}\left(X^{*}, \infty\right) = \operatorname*{ess\,sup}_{s \in L^{0}} R(X^{*}, s) \quad \text{for some } m \in \bar{L}^{0},\ X^{*} \in \mathcal{K}^{\circ}\Big)\\ &\quad\Longrightarrow \Big(m \geq R^{-}\left(Y^{*}, \infty\right) = \operatorname*{ess\,sup}_{s \in L^{0}} R(Y^{*}, s),\ \text{ and for all } Y^{*} \in \mathcal{K}^{\circ}\Big).\end{aligned} \tag{1.52}$$

Taking in the last implication $m = R^{-}(X^{*}, \infty)$, we get that $R^{-}(X^{*}, \infty) \geq R^{-}(Y^{*}, \infty)$ for any Y^{*}. Applying the equivalence consequently to $m = R^{-}(Y^{*}, \infty)$, we conclude that

$$R^{-}\left(X^{*}, \infty\right) = R^{-}(Y^{*}, \infty) \quad \text{for all } X^{*}, Y^{*} \in \mathcal{K}^{\circ}. \tag{1.53}$$

Clearly, if (1.53) holds true, then implication (1.52) also holds true, and hence (1.52) is equivalent to (1.52). Thus, π satisfies (d) if and only if R satisfies (iv) which completes the proof. □

1.6.4 Proof of Proposition 1.11

Before proving Proposition 1.11, we first give the definition of the conditional characteristic function, followed by Proposition 1.52 that contains some relevant properties of the conditional characteristic function.

Definition 1.51. Let $\mathcal{C}$ be a σ-stable subset of $\mathcal{X}$. For $X \in \mathcal{X}$ we define $A(X) = \operatorname{ess\,sup}\{B \in \mathscr{G} : 1_B X \in 1_B \mathcal{C}\}$. The function $\chi_{\mathcal{C}} : \mathcal{X} \to \bar{L}^{0}$ given by

$$\chi_{\mathcal{C}}\left(X\right) = -1_{A^{c}(X)}\infty = \begin{cases} 0 & \text{on } A(X) \\ -\infty & \text{on } A^{c}(X) \end{cases}, \quad X \in \mathcal{X},$$

is called the conditional characteristic function of $\mathcal{C}$.

Note that the conditional characteristic function is a mapping from $\mathcal{X}$ to $\bar{L}^{0}$.

Proposition 1.52. *Let $\mathcal{C}$ be a σ-stable set. Then, $\chi_{\mathcal{C}}$ is a local function. Furthermore,*

- *$\mathcal{C}$ is nonempty if and only if $\chi_{\mathcal{C}}$ is proper;*
- *$\mathcal{C}$ is monotone if and only if $\chi_{\mathcal{C}}$ is monotone;*
- *$\mathcal{C}$ is convex if and only if $\chi_{\mathcal{C}}$ is concave;*

- *$\mathcal{C}$ is a cone if and only if $\chi_{\mathcal{C}}$ is positive homogeneous;*
- *$\mathcal{C}$ is closed if and only if $\chi_{\mathcal{C}}$ is upper semicontinuous.*

Proof. For $B \in \mathscr{G}$ and $X \in \mathcal{C}$, since $\mathcal{C}$ is σ-stable, it holds

$$\begin{aligned} B \cap A(X) &= \operatorname{ess\,sup}\{\widetilde{B} \cap B : \widetilde{B} \in \mathscr{G} \text{ and } 1_{\widetilde{B}} X \in 1_{\widetilde{B}}\mathcal{C}\} \\ &= B \cap \operatorname{ess\,sup}\left\{\widetilde{B} \in \mathscr{G} : 1_{\widetilde{B}} 1_B X \in 1_{\widetilde{B}} 1_B \mathcal{C}\right\} = B \cap A(1_B X). \end{aligned}$$

This implies that $B \cap A^c(X) = B \cap A^c(1_B X)$, and hence, $1_B \chi_{\mathcal{C}}(1_B X) = 1_B \chi_{\mathcal{C}}(X)$, and therefore $\chi_{\mathcal{C}}$ is local. By definition, $\chi_{\mathcal{C}} < +\infty$. Reversely, $A(X)$ is of measure zero for every X if and only if $\mathcal{C}$ is the empty set, therefore $\chi_{\mathcal{C}}$ is proper if and only if $\mathcal{C}$ is nonempty. The monotonicity of $\mathcal{C}$ implies the monotonicity of $\chi_{\mathcal{C}}$ is immediate by the definition of $A(X)$, $X \in \mathcal{X}$. Since $X \in \mathcal{C}$ if and only if $\chi_{\mathcal{C}}(X) = 0$, the converse implication also follows. Using σ-stability of $\mathcal{C}$, it can be showed that $A(\lambda X + (1-\lambda)Y) \supseteq (A(X) \cap A(Y))$ if and only if $\mathcal{C}$ is convex, and so $\chi_{\mathcal{C}}$ is a concave function if and only if $\mathcal{C}$ is convex. Similarly one proves that $\mathcal{C}$ is a cone if and only if $\chi_{\mathcal{C}}$ is positive homogeneous.

Finally, if $\mathcal{C} = \emptyset$, clearly $\chi_{\mathcal{C}}$ is upper semicontinuous. Otherwise, note that

$$\{X \in \mathcal{X} : \chi_{\mathcal{C}}(X) \geq m\} = \begin{cases} 1_{\{m > -\infty\}}\mathcal{C} + 1_{\{m = -\infty\}}\mathcal{X} & \text{if } m \leq 0, \\ \emptyset & \text{otherwise,} \end{cases}$$

which is a closed set for every $m \in \bar{L}^0$ if and only if $\mathcal{C}$ is closed. □

Proof of Proposition 1.11. If $\mathcal{C} = \emptyset$, then $\pi \equiv \infty$ fulfills all required conditions. Hence, we will consider the case $\mathcal{C} \neq \emptyset$.

Step 1: We first assume that $\mathcal{K} = \{0\}$, so that $\mathcal{K}^\circ = \mathcal{X}^*$. We start with the existence of π. Since, $\mathcal{C} \neq \emptyset$, by Proposition 1.52, the conditional characteristic function $\chi_{\mathcal{C}}$ is a local, proper, concave and upper semicontinuous function. Using the definition of $\chi_{\mathcal{C}}$, we deduce that its concave conjugate $\chi^\star_{\mathcal{C}}(X^*) := \operatorname{ess\,inf}_{X \in \mathcal{X}} \{\langle X^*, X\rangle - \chi_{\mathcal{C}}(X)\}$ can be also represented as follows

$$\chi^\star_{\mathcal{C}}(X^*) = \operatorname*{ess\,inf}_{X \in \mathcal{C}} \langle X^*, X\rangle, \quad X^* \in \mathcal{X}^*. \tag{1.54}$$

Indeed, since $X \in \mathcal{C}$ if and only if $\chi_{\mathcal{C}}(X) = 0$, it clearly follows that $\chi^\star_{\mathcal{C}}(X^*) \leq \operatorname{ess\,inf}_{X \in \mathcal{C}} \langle X^*, X\rangle$. Suppose now that there exists $X_0 \in \mathcal{X}$ such that

$$1_A \langle X^*, X_0\rangle - 1_A \chi_{\mathcal{C}}(X_0) < 1_A \operatorname*{ess\,inf}_{\mathcal{C}} \langle X^*, X\rangle \tag{1.55}$$

on some set A. Note that by locality, the definition of $\chi_{\mathcal{C}}$ and the fact that $\mathcal{C} \neq \emptyset$, we have that $1_A \chi^\star_{\mathcal{C}}(X^*) = 1_A \chi^\star_{\mathcal{C}}(1_A X^*)$, $1_A \operatorname{ess\,inf}_{\mathcal{C}} \langle X^*, X\rangle = \operatorname{ess\,inf}_{X \in 1_A \mathcal{C}} \langle 1_A X^*, X\rangle$

and $1_A\chi_{\mathcal{C}}(X) = \chi_{1_A\mathcal{C}}(1_A X)$. However, the strict inequality in (1.55) implies that $1_A\chi_{\mathcal{C}}(X_0) = \chi_{1_A\mathcal{C}}(1_A X_0) > -\infty$, that is $\chi_{1_A\mathcal{C}}(1_A X_0) = 0$. Hence

$$\begin{aligned} 1_A\langle X^*, X_0\rangle - 1_A\chi_{\mathcal{C}}(X_0) &= 1_A\langle X^*, 1_A X_0\rangle - \chi_{1_A\mathcal{C}}(1_A X_0) \geq 1_A \operatorname*{ess\,inf}_{X\in 1_A\mathcal{C}}\langle X^*, X\rangle \\ &= 1_A \operatorname*{ess\,inf}_{X\in\mathcal{C}}\langle X^*, X\rangle \end{aligned}$$

showing together with (1.55) that A is a set of null measure.

Note that by Proposition 1.42, $\chi_{\mathcal{C}}^{\star}$ is upper semicontinuous and concave and clearly positive homogeneous. Furthermore, since $\mathcal{C} \neq \emptyset$, in view of (1.55), it follows that $\chi_{\mathcal{C}}^{\star} < \infty$ and therefore maximal invariant.

By the conditional Fenchel-Moreau theorem (compare Proposition 1.42(3) or [43, Theorem 3.8]), we have that

$$\chi_{\mathcal{C}}\left(X\right) = \chi_{\mathcal{C}}^{\star\star}\left(X\right) := \operatorname*{ess\,inf}_{X^*\in\mathcal{X}^*}\{\langle X^*, X\rangle - \chi_{\mathcal{C}}^{\star}\left(X^*\right)\}.$$

Hence, by the definition of $\chi_{\mathcal{C}}$ and (1.54) it follows that

$$\begin{aligned} X\in\mathcal{C} \quad &\Longleftrightarrow \quad 0 \leq \chi_{\mathcal{C}}(X) = \operatorname*{ess\,inf}_{X^*\in\mathcal{X}^*}\{\langle X^*, X\rangle - \chi_{\mathcal{C}}^{\star}\left(X^*\right)\} \\ &\Longleftrightarrow \quad \langle X^*, X\rangle \geq \chi_{\mathcal{C}}^{\star}(X^*) = \operatorname*{ess\,inf}_{Y\in\mathcal{C}}\langle X^*, Y\rangle, \quad \text{for all } X^*\in\mathcal{X}^*. \end{aligned}$$

Thus, the function

$$\pi\left(X^*\right) := \operatorname*{ess\,inf}_{X\in\mathcal{C}}\langle X^*, X\rangle, \quad X^*\in\mathcal{X}^*,$$

fulfills relation (1.5) and the conditions (a) to (c).

Step 2: As for the uniqueness of π, let $\pi^1, \pi^2 : \mathcal{X}^* \to \bar{L}^0$ fulfill the conditions (a) to (c) and relation (1.5). We will still assume that $\mathcal{K} = \{0\}$. If $\pi^1\left(X^*\right) = \infty$ for some $X^*\in\mathcal{X}^*$, then by relation (1.5), it follows that $\mathcal{C} = \emptyset$ which implies $\pi^2(Y^*) = \infty$ for some $Y^*\in\mathcal{X}^*$. Since both are maximal invariant, it follows that $\pi^1 = \pi^2 = \infty$. Now suppose $\pi^i < \infty$.[29]

We claim that, for $i = 1, 2$,

$$\Big(1_B\pi^i(X^*) = -1_B\infty \text{ for all } X^*\in\mathcal{X}^*\Big) \quad \Longleftrightarrow \quad 1_B\mathcal{C} = 1_B\mathcal{X}. \tag{1.56}$$

Indeed, if $1_B\pi^i(X^*) = -1_B\infty$, for all $X^*\in\mathcal{X}^*$, then

$$1_B\langle X, X^*\rangle \geq 1_B\pi^i(X^*), \tag{1.57}$$

[29] Note that since π^i is maximal invariant then we only need to consider two cases: $\pi^i = \infty$ and $\pi^i < \infty$, $i = 1, 2$.

for all $X \in \mathcal{X}$ and all $X^* \in \mathcal{X}^*$. Since $\mathcal{C} \neq \emptyset$, we choose any $Y \in \mathcal{C}$. By (1.5), we get that $1_{B^c}\langle Y, X^*\rangle \geq 1_{B^c}\pi^i(X^*)$, for all $X^* \in \mathcal{X}^*$, which combined with (1.57), and by locality, gives us

$$\langle 1_B X + 1_{B^c} Y, X^*\rangle \geq \pi^i(X^*), \quad \text{for all } X^* \in \mathcal{X}^*,$$

and thus $Z = 1_B X + 1_{B^c} Y \in \mathcal{C}$. Moreover, $1_B X = 1_B Z$, and hence since X was arbitrary in $\mathcal{X}$, we conclude that $1_B\mathcal{X} \subseteq 1_B\mathcal{C}$. The inclusion $1_B\mathcal{X} \supseteq 1_B\mathcal{C}$ is obvious, and thus $1_B\mathcal{X} = 1_B\mathcal{C}$.

Assume that $1_B\mathcal{X} = 1_B\mathcal{C}$ and that there exist $X_0^* \in \mathcal{X}^*,\ B_1 \subseteq B$ such that $1_{B_1}\pi^i(X_0^*) > -1_{B_1}\infty$ on B_1. Choose $X_0 \in \mathcal{X}$ such that $\langle X_0, X_0^*\rangle < 0$ on B_1. Then, for a sufficiently large $\lambda_0 \in L^0_{++}$, we have that $\langle \lambda_0 X_0, X_0^*\rangle < \pi^i(X_0^*)$ on B_1, and hence $1_{B_1}\lambda_0 X_0 \notin 1_{B_1}\mathcal{C}$. However, $1_{B_1}\lambda_0 X_0 \in 1_{B_1}\mathcal{X} = 1_{B_1}\mathcal{C}$, which yields a contradiction. Thus, the equivalence (1.56) is established.

Next, define the sets

$$A^i := \operatorname{ess\,sup}\left\{B \in \mathscr{F} : 1_B\pi^i(X^*) = -1_B\infty \text{ for all } X^* \in \mathcal{X}^*\right\},\ i = 1, 2.$$

By (1.56), we get that $A_1 = A_2$. Note that on the set $A^1 = A^2$, the functions π^1 and π^2 coincides and are both equal to $-\infty$. Define $\tilde{\pi}^i = 1_A\pi^i$, where $A := (A^1)^c = (A^2)^c$. These functions are concave, upper semicontinuous and local since both the π^i are so, and proper by the definition of A. Due to the conditional Fenchel-Moreau theorem we obtain

$$\tilde{\pi}^i\left(X^*\right) = \operatorname*{ess\,inf}_{X \in \mathcal{X}}\left\{\langle X^*, X\rangle - \tilde{\pi}^{i,\star}\left(X\right)\right\}, \quad X^* \in \mathcal{X}^*, \tag{1.58}$$

where

$$\tilde{\pi}^{i,\star}\left(X\right) = \operatorname*{ess\,inf}_{X^* \in \mathcal{X}^*}\left\{\langle X^*, X\rangle - \tilde{\pi}^i(X^*)\right\}, \quad X \in \mathcal{X}.$$

Since $\tilde{\pi}^i$ is positively homogeneous, and $\tilde{\pi}^{i,\star}$ is proper, we have that $\tilde{\pi}^{i,\star}$ can only take the values 0 or $-\infty$. Therefore,

$$\tilde{\pi}^{\star,i}(X) = 0 \quad \Longleftrightarrow \quad \langle X^*, X\rangle \geq \pi^i(X^\star) \text{ for all } X^* \in \mathcal{X}^* \quad \Longleftrightarrow \quad X \in 1_A\mathcal{C}.$$

Hence $\tilde{\pi}^{\star,1} = \tilde{\pi}^{2,\star}$, which together with equation (1.58) implies that $\tilde{\pi}^i = \tilde{\pi}^2$. Thus, $\pi^1 = \pi^2$.

Step 3: Finally, let us consider the case where $\mathcal{K} \neq \{0\}$, that is $\mathcal{K}^\circ \neq \mathcal{X}^*$. As we already showed in Step 1, the function $\pi : \mathcal{X}^* \to \bar{L}^0$ given by

$$\pi(X^*) = \operatorname*{ess\,inf}_{X \in \mathcal{C}}\langle X^*, X\rangle, \quad X^* \in \mathcal{X}^*,$$

satisfies conditions (a)-(c), hence its restriction on $\mathcal{K}^\circ$ satisfies (a)-(c). Taking into account the uniqueness proved in Step 2, the proof will be complete if we show that $\pi : \mathcal{K}^\circ \to \bar{L}^0$ fulfills (1.5).

First, we will show that for any $X^* \in \mathcal{X}^*$, we have that

$$\pi(X^*) = -\infty \quad \text{on } A^c_{X^*},$$

where $A_{X^*} = \operatorname{ess\,sup}\{B \in \mathscr{G} : 1_B X^* \in \mathcal{K}^\circ\}$. Indeed, by definition of the polar cone and A_{X^*}, it follows that there exists $Y \in \mathcal{K}$ such that

$$\langle X^*, Y\rangle < 0, \qquad \text{on } A^c_{X^*}.$$

Choose $\hat{X} \in \mathcal{C}$; by monotonicity of $\mathcal{C}$, we get that $\hat{X} + \lambda Y \in \mathcal{C}$ for every $\lambda > 0$. Hence,

$$\pi(X^*) = \operatorname*{ess\,inf}_{X \in \mathcal{C}} \langle X^*, X\rangle \leq \langle X^*, \hat{X}\rangle + \lambda\langle X^*, Y\rangle, \qquad \text{for every } \lambda > 0.$$

Hence, letting λ going to ∞, and taking into account that $\langle X^*, Y\rangle < 0$ on $A^c_{X^*}$, we conclude that $\pi(X^*)$ is equal to $-\infty$ on $A^c_{X^*}$.

Next, define $\widetilde{X}^* := 1_A X^*$, and note that by the definition of A_{X^*}, we have that $\widetilde{X}^* \in \mathcal{K}^\circ$. Since $\pi(X^*) = -\infty$ on $A^c_{X^*}$, by locality and the fact that $\pi(0) = 0$, it follows that

$$\langle X^*, X\rangle \geq \pi(X^*) \quad \Longleftrightarrow \quad \langle \widetilde{X}^*, X\rangle = 1_{A_{X^*}}\langle X^*, X\rangle \geq 1_{A_{X^*}}\pi(X^*) = \pi(\widetilde{X}^*). \tag{1.59}$$

Note that by locality and the definition of A_{X^*} and $\widetilde{X}^*$, we have that

$$\mathcal{K}^\circ = \{Y^* \in \mathcal{X}^* : \text{ there exists } X^* \in \mathcal{X}^* \text{ such that } Y^* = 1_{A_{X^*}} X^*\}.$$

Using this and (1.59), we conclude that

$$X \in \mathcal{C} \quad \Longleftrightarrow \quad \langle X^*, X\rangle \geq \pi(X^*) \quad \text{for all } X^* \in \mathcal{K}^\circ.$$

This completes the proof. $\square$

2 Ekeland's Variational Principle in L^0-Metric Modules

In this chapter, we develop an Ekeland's variational principle for L^0-metric modules. The variational principle by Ekeland was introduced 1974 in [39]. The main theorem, which we will call Ekeland's theorem, deals with functions on an arbitrary complete metric space (X, d). Let f be a lower semicontinuous function from X to $\mathbb{R} \cup \{+\infty\}$ attaining at least one real value and fulfilling $\inf f > -\infty$. Then, for y with $f(y) \in \mathbb{R}$ there exists x with $f(x) \in \mathbb{R}$ such that $f(x)+d(y,x) \leq f(y)$ and for every $z \neq x$ it holds that $f(x) < f(z)+d(z,x)$. A corollary of this theorem, which is also called ε-variational principle (compare [5]), is the following. Let f fulfill the same properties as before. Let further $y \in X$ fulfill $\inf f \leq f(y) \leq \inf f + \varepsilon$ for some $\varepsilon > 0$. Then, for every $\lambda > 0$ there exists $x \in X$ such that

(1) $f(x) \leq f(y)$;

(2) $d(x, y) \leq \lambda$;

(3) for every $z \neq x$ it holds that $f(x) - f(z) < (\varepsilon/\lambda)d(x, z)$.

We see that in the theorem and the corollary an interaction of the metric d and the function f is described. Moreover, property (3) shows that x is the unique minimizer of the function $z \mapsto f(z) + (\varepsilon/\lambda)d(x, z)$. In the case $\lambda = 1$, this x is also called ε-minimizer. For the infimum of f and for every $\varepsilon > 0$ there exists some y fulfilling $\inf f \leq f(y) \leq \inf f + \varepsilon$. Hence, the theorem can be applied to minimization problems without using analytic methods. We will develop Ekeland's variational principle for L^0-modules. The metric will therefore map to L^0_+. In [59] it was examined which of the properties Ekeland assumed are necessary and which can be dropped. The generalizations provided there are also transfered to the L^0-setting. Hence, considering the trivial σ-algebra $\mathcal{F} = \{\emptyset, \Omega\}$ we recover the results of [59] which are still more general as Ekeland's variational principle.

This chapter is organized as follows. In the first section, we introduce the basic notions. We define L^0-metric modules, local preorders and decreasing local sequences. In the second section, we prove a result on minimal and invariant points for local preorders. This is one fundamental result and we show its equivalence to further results, for instance to a fixed point theorem of a local set-valued map. In the third section, we prove Ekeland's variational principle for L^0-metric modules. We define local order monoids and local premetrics mapping to it and prove Ekeland's theorem in our setting. Caristi [17] observed that his fixed point theorem is an equivalent formulation of Ekeland's variational principle. We will do the same in our setting by showing that Ekeland's theorem implies both the Kirk-Caristi fixed point theorem and the Takahashi's minimization theorem.

2.1 Preliminaries

We consider $L^0 = L^0(\Omega, \mathcal{F}, P)$. In $\mathcal{F}$, sets A and B for which $P((A\backslash B) \cup (B\backslash A)) = 0$ are identified. For $\mathcal{G} \subseteq \mathcal{F}$, we denote by $\vee\, \mathcal{G}$ and by $\wedge\, \mathcal{G}$ the supremum and the infimum of $\mathcal{G}$, respectively, with respect to P-almost sure inclusion. Throughout this chapter, $\mathcal{X}$ denotes a σ-stable L^0-module.

Definition 2.1. A local function $d\colon \mathcal{X} \times \mathcal{X} \to L^0_+$ is called an L^0-metric if

(i) $d(X, Y) = 0$ if and only if $X = Y$.

(ii) $d(X, Y) = d(Y, X)$ for all $X, Y \in \mathcal{X}$.

(iii) $d(X, Y) \leq d(X, Y) + d(Y, Z)$ for all $X, Y, Z \in \mathcal{X}$.

The pair $(\mathcal{X}, d)$ is called L^0-metric module. A local sequence in $\mathcal{X}$ is a local function from $\mathbb{N}(\mathcal{F})$ to $\mathcal{X}$, $N \mapsto X_N$ and is denoted by $(X_N)_{N\in\mathbb{N}(\mathcal{F})}$ or just (X_N). Further, $(X_N)_{N\in\mathbb{N}(\mathcal{F})}$ is called a Cauchy sequence if for any $\varepsilon \in L^0_{++}$ there exists $N_0 \in \mathbb{N}(\mathcal{F})$ such that $d(X_{N_1}, X_{N_2}) \leq \varepsilon$ for all $N_1, N_2 \geq N_0$. Moreover, $(X_N)_{N\in\mathbb{N}(\mathcal{F})}$ converges to X, denoted by $X_N \to X$, if for every $\varepsilon \in L^0_{++}$ there exists $N_0 \in \mathbb{N}(\mathcal{F})$ such that $d(X_N, X) < \varepsilon$ for all $N \geq N_0$. In this case, X is said to be the limit of $(X_N)_{N\in\mathbb{N}(\mathcal{F})}$.

Every local sequence can be seen as a net with index set $\mathbb{N}(\mathcal{F})$. The definition of convergence coincides with the topology generated by the collection $\{B_\varepsilon(X) : X \in \mathcal{X}, \varepsilon \in L^0_{++}\}$ where $B_\varepsilon(X) = \{Y \in \mathcal{X} : d(X, Y) < \varepsilon\}$ (compare [43]). Note that L^0 itself is an L^0-metric module for the L^0-metric $d(X, Y) := |X - Y|$. It is in this sense that convergence of local sequences in L^0 has to be understood.

Example 2.2. For a sequence $(X_n)_{n\in\mathbb{N}} \subseteq \mathcal{X}$ and $N \in \mathbb{N}(\mathcal{F})$, we canonically define

$$X_N := \sum_{n\in\mathbb{N}} 1_{\{N=n\}} X_n.$$

This $(X_N)_{N\in\mathbb{N}(\mathcal{F})}$ is the local sequence corresponding to the sequence $(X_n)_{n\in\mathbb{N}}$. Note that in this construction X_N is again an element of $\mathcal{X}$ because $\{N = n\} = \{\omega : N(\omega) = n\}_{n\in\mathbb{N}}$ is a partition. Reversely, considering a local sequence $(X_N)_{N\in\mathbb{N}(\mathcal{F})}$, then X_n, for $n \in \mathbb{N}$, denotes the element X_N with $N = 1_\Omega n$.

Definition 2.3. A local relation $\preceq$ on $\mathcal{X}$ is a binary relation on $\mathcal{X}$ such that $\mathcal{R} := \{(Y,X) : Y \preceq X\}$ is σ-stable. A local preorder is a local relation which is reflexive, that is $X \preceq X$ for every $X \in \mathcal{X}$, and transitive, that is $X \preceq Y \preceq Z$ implies $X \preceq Z$ for every $X, Y, Z \in \mathcal{X}$. For $\preceq$ we denote the lower level set of $X \in \mathcal{X}$ by $L(X) = \{Y \in \mathcal{X} : Y \preceq X\}$.

Given a relation $\preceq$, $\mathcal{R}$ always denotes the corresponding set as in the previous definition. We notice that $L(X)$ is always σ-stable in $\mathcal{X}$. Indeed, consider a partition $(A_n)_{n\in\mathbb{N}}$ and $(Y_n)_{n\in\mathbb{N}} \subseteq L(X)$. Since $\preceq$ is local, it holds that $(\sum_{n\in\mathbb{N}} 1_{A_n} Y_n, \sum_{n\in\mathbb{N}} 1_{A_n} X) \in \mathcal{R}$. As $\sum_{n\in\mathbb{N}} 1_{A_n} X = X$, this implies $\sum_{n\in\mathbb{N}} 1_{A_n} Y_n \preceq X$ and hence $\sum_{n\in\mathbb{N}} 1_{A_n} Y_n \in L(X)$.

Definition 2.4. Let $\preceq$ be a local preorder on $\mathcal{X}$. A local sequence $(X_N)_{N\in\mathbb{N}(\mathcal{F})}$ is called decreasing if $X_M \preceq X_N$ for every $M, N \in \mathbb{N}(\mathcal{F})$ with $M \geq N$.

Remark 2.5. If $(X_n)_{n\in\mathbb{N}}$ is a decreasing sequence, that is $X_n \preceq X_{n+1}$ for all $n \in \mathbb{N}$, then the corresponding local sequence is also decreasing. Indeed, as the relation is transitive, for every decreasing sequence $(X_n)_{n\in\mathbb{N}}$ it holds that $X_m \preceq X_n$ for every $m, n \in \mathbb{N}$, $m \geq n$. This implies $X_M \preceq X_N$ for every $M, N \in \mathbb{N}(\mathcal{F}), M \geq N$. To show this, let $M \geq N$. For P-almost every ω it holds that if $N(\omega) = n$ for $n \in \mathbb{N}$, then there exists $m \in \mathbb{N}, m \geq n$ with $M(\omega) = m$. For $A_n = \{N = n\}$, $n \in \mathbb{N}$ and $B_m = \{M = m\}$, $m \in \mathbb{N}$, we define $(C_k)_{k\in\mathbb{N}}$ to be the refinement of the partitions $(A_n)_{n\in\mathbb{N}}$ and $(B_m)_{m\in\mathbb{N}}$. Hence, on every C_k it holds $N \equiv n_k$ and $M \equiv m_k$ with $m_k \geq n_k$. Thus, $X_N = \sum_{k\in\mathbb{N}} 1_{\{N=n_k\}} X_{n_k}$ and $X_M = \sum_{k\in\mathbb{N}} 1_{\{N=n_k\}} X_{m_k}$. As every $X_{m_k} \preceq X_{n_k}$, the assertion follows by using locality of $\preceq$.

Moreover, the above shows that it is sufficient to demand $X_{N+1} \preceq X_N$ for any $N \in \mathbb{N}(\mathcal{F})$ to characterize decreasing local sequences.

Definition 2.6. Let $(\mathcal{X}, d)$ be an L^0-metric module with a local preorder $\preceq$ on it. Then $\mathcal{X}$ is called $\preceq$–complete if every decreasing Cauchy sequence has a limit in $\mathcal{X}$. Further, we call $\mathcal{X}$ lower closed if for every decreasing local sequence $(X_N)_{N\in\mathbb{N}(\mathcal{F})}$ converging to $X \in \mathcal{X}$ it holds that $X \preceq X_N$ for any $N \in \mathbb{N}(\mathcal{F})$. A local preorder $\preceq$ is called regular if for every decreasing local sequence $(X_N)_{N\in\mathbb{N}(\mathcal{F})} \subseteq \mathcal{X}$ the local sequence $(d(X_N, X_{N+1}))_{N\in\mathbb{N}(\mathcal{F})} \subseteq L^0$ converges to zero.

Example 2.7. A prototype of a regular, lower closed local preorder is the following. Consider a local, lower semicontinuous function $f : \mathcal{X} \to L^0$ bounded from below, that is there exists $Z \in L^0$ such that $f(X) \geq Z$ for every $X \in \mathcal{X}$. Let further $\varepsilon \in L^0_{++}$ and define

$$X \precsim Y \iff f(X) + \varepsilon d(X, Y) \leq f(Y). \tag{2.1}$$

This relation is local, lower closed and regular. Concerning the regularity, note that $X \precsim Y$ implies $f(Y) \geq f(X)$, since d maps to L^0_+ and it even holds $f(Y) > f(X)$ on $(\vee\{A \in \mathcal{F} : 1_A X \precsim 1_A Y\})^c$. Hence, if $(X_N)_{N\in\mathbb{N}(\mathcal{F})}$ is decreasing (with respect to $\precsim$), then

$(f(X_N))_{N\in\mathbb{N}(\mathcal{F})}$ is decreasing (with respect to $\leq$) and hence also $(d(X_N, X_{N+1}))_{N\in\mathbb{N}(\mathcal{F})}$ by (2.1). Supposing that $(d(X_N, X_{N+1}))_{N\in\mathbb{N}(\mathcal{F})}$ does not converge yields a contradiction to (2.1).

2.2 Minimal and Invariant Points of Local Preorders

To derive the main theorem in [39], Ekeland first proved a result on maximal points for relations. In this section, we derive a minimal point result for local preorders which we will use later on to obtain our main theorem. Throughout this section, let $(\mathcal{X}, d)$ be an L^0-metric module. Given a relation $\preceq$, we denote $X \sim Y$ if $X \preceq Y$ and $Y \preceq X$. A relation is called antisymmetric if $X \sim Y$ implies $X = Y$. For a reflexive relation $X \sim Y$ and $X = Y$ are hence equivalent.

Lemma 2.8. *A regular local preorder is antisymmetric.*

Proof. Let $X \sim Y$. Consider the sequence $(X_n)_{n\in\mathbb{N}} = (X, Y, X, \dots)$ and the corresponding local sequence $(X_N)_{N\in\mathbb{N}(\mathcal{F})}$ which is decreasing. Hence, by regularity it follows that $d(X_N, X_{N+1}) \to 0$ which is only possible if $d(X, Y) = 0$. As d is an L^0-metric, it follows that $X = Y$. □

Definition 2.9. An element $X \in \mathcal{X}$ is called minimal with respect to the local preorder $\preceq$ if $Y \preceq X$ implies that $X \sim Y$. We further denote

$$S(\mathcal{X}) = \{\mathcal{Y} \subseteq \mathcal{X} : \mathcal{Y} \text{ is } \sigma\text{-stable}\}.$$

A set-valued map T is a local function $T\colon \mathcal{X} \to S(\mathcal{X})$. An element $\bar{X} \in \mathcal{X}$ is said to be a fixed point of T if $\bar{X} \in T(\bar{X})$. An element $\bar{X} \in \mathcal{X}$ is said to be an invariant point of T if $T(\bar{X}) = \{\bar{X}\}$.

If $\preceq$ is regular, a minimal element X fulfills $L(X) = \{X\}$ due to antisymmetry. Furthermore, the set $S(\mathcal{X})$ is σ-stable. Indeed, given a sequence $(\mathcal{X}_n)_{n\in\mathbb{N}}$ of σ-stable subsets of $\mathcal{X}$ and a partition $(A_n)_{n\in\mathbb{N}}$ the set $\sum_{n\in\mathbb{N}} 1_{A_n}\mathcal{X}_n$ itself is σ-stable.

The next theorem contains three implications of a regular, lower closed local preorder in a complete L^0-metric module. Implication (I1) states that starting with an arbitrary $X \in \mathcal{X}$, there always exists a minimal element $\bar{X}$ in its lower level set. Implication (I2) is close to a transfer of Theorem 3.1 in [27] to the L^0-setting and shows the existence of a fixed point and an invariant point for set-valued maps fulfilling certain conditions. Oettli and Théra [65] proved an equivalent formulation of Ekeland's principle. A generalization of this result within the L^0-setting is given by implication (I3).

Theorem 2.10. *Let $\preceq$ be a regular, lower closed, local preorder on $\mathcal{X}$ such that $(\mathcal{X}, d)$ is $\preceq$–complete. Then, the following holds:*

(I1) For each $Y \in \mathcal{X}$ there exists $\bar{X} \in \mathcal{X}$ such that

$$\bar{X} \preceq Y, \quad \text{and} \quad X' \preceq \bar{X} \text{ implies } X' = \bar{X}.$$

(I2) Let $T\colon \mathcal{X} \to S(\mathcal{X})$ be a set-valued map. If $T(X) \cap L(X)$ is nonempty, for every $X \in \mathcal{X}$, then T has a fixed point. If T satisfies $T(X) \subseteq L(X)$, for every $X \in \mathcal{X}$, then T has an invariant point.

(I3) Let $\mathcal{M} \in S(\mathcal{X})$ be such that for every $X \in L(X_0)\backslash\mathcal{M}$ there exists $X' \in L(X)\backslash\{X\}$. Then, there exists $\bar{X} \in L(X_0) \cap \mathcal{M}$.

Proof. Proof of (I1): Starting with $X_1 := Y$, we define a sequence $(X_n)_{n\in\mathbb{N}}$ as follows. Given X_n, we denote by $A_n := \vee\{A \in \mathcal{F} : 1_A \operatorname{ess\,sup}_{X\in L(X_n)} d(X, X_n) \in L^0\}$. As d is a local function, A_n is attained, that is $1_{A_n} \operatorname{ess\,sup}_{X\in L(X_n)} d(X, X_n) \in L^0$. Reversely, it holds that $1_{A_n^c} \operatorname{ess\,sup}_{X\in L(X_n)} d(X, X_n) = 1_{A^c}\infty$ meaning it exceeds every $Y \in L^0_{++}$ on A_n^c. Pick $X_{n+1} \in L(X_n)$ fulfilling

$$d(X_{n+1}, X_n) \geq 1_{A_n}\left[\operatorname*{ess\,sup}_{X\in L(X_n)} d(X, X_n) - \frac{1}{n}\right] + 1_{A_n^c}.$$

Consider the local sequence $(X_N)_{N\in\mathbb{N}(\mathcal{F})}$ corresponding to $(X_n)_{n\in\mathbb{N}}$, defined by $X_N := \sum_{n\in\mathbb{N}} 1_{\{N=n\}} X_n$. By setting

$$A_N := \vee\left\{A : 1_A \operatorname*{ess\,sup}_{X\in L(X_N)} d(X, X_N) \in L^0\right\}$$

it holds that

$$d(X_{N+1}, X_N) \geq 1_{A_N}\left[\operatorname*{ess\,sup}_{X\in L(X_N)} d(X, X_N) - \frac{1}{N}\right] + 1_{A_N^c}.$$

Indeed, first note that with $B_n := \{N = n\}$ it holds $A_N = \cup_{n\in\mathbb{N}}(B_n \cap A_n)$ due to

$$\begin{aligned}
A_N &= \vee\left\{A : 1_A \operatorname*{ess\,sup}_{X\in L(X_N)} d(X, X_N) \in L^0\right\} \\
&= \vee\left\{A : 1_A \operatorname*{ess\,sup}_{X\in\sum_{n\in\mathbb{N}} 1_{B_n} L(X_n)} d(X, X_n) \in L^0\right\} \\
&= \bigcup_{n\in\mathbb{N}}\left\{A \cap B_n : 1_{A\cap B_n} \operatorname*{ess\,sup}_{X\in L(X_n)} d(X, X_n) \in L^0\right\} \\
&= \bigcup_{n\in\mathbb{N}} B_n \cap A_n.
\end{aligned}$$

We used the σ-stability of the lower level sets and that the relation is reflexive, thereby $X_n \in L(X_n)$, which allows us to obtain elements defined on whole Ω if necessary.

Moreover, we obtain

$$
\begin{aligned}
d(X_{N+1}, X_N) &= \sum_{n\in\mathbb{N}} 1_{B_n} d(X_{n+1}, X_n) \\
&\geq \sum_{n\in\mathbb{N}} 1_{B_n} \left(1_{A_n} \left[\operatorname*{ess\,sup}_{X\in L(X_n)} d(X, X_n) - \frac{1}{n} \right] + 1_{A_n^c} \right) \\
&= \sum_{n\in\mathbb{N}} 1_{B_n\cap A_n} \left[\operatorname*{ess\,sup}_{X\in \sum_{n\in\mathbb{N}} 1_{B_n} L(X_n)} d\left(X, \sum_{n\in\mathbb{N}} 1_{B_n} X_n\right) - \sum_{n\in\mathbb{N}} 1_{B_n} \frac{1}{n} \right] + \sum_{n\in\mathbb{N}} 1_{B_n\cap A_n^c} \\
&= \sum_{n\in\mathbb{N}} 1_{B_n\cap A_n} \left[\operatorname*{ess\,sup}_{X\in L(X_N)} d(X, X_N) - \frac{1}{N} \right] + 1_{A_N^c} \\
&= 1_{A_N} \left[\operatorname*{ess\,sup}_{X\in L(X_N)} d(X, X_N) - \frac{1}{N} \right] + 1_{A_N^c}.
\end{aligned}
$$

We used that $[\cup_{n\in\mathbb{N}}(B_n \cap A_n^c)] \cup [\cup_{n\in\mathbb{N}}(B_n \cap A_n)] = \cup_{n\in\mathbb{N}} B_n = \Omega$ as $(B_n)_{n\in\mathbb{N}}$ is a partition and therefore $\cup_{n\in\mathbb{N}}(B_n \cap A_n^c) = A_N^c$.

Furthermore, working with a local preorder, it follows that $X_{N+1} \in L(X_N)$ for all $N \in \mathbb{N}(\mathcal{F})$. Since $\preceq$ is transitive and $X_{N+1} \preceq X_N$, it holds that $L(X_{N+1}) \subseteq L(X_N)$. There exists a local sequence $(Y_K)_{K\in\mathbb{N}(\mathcal{F})}$ in $L(X_{N+1})$ such that $(d(Y_K, X_{N+1}))_{K\in\mathbb{N}(\mathcal{F})}$ exceeds any element of L^0 on A_{N+1}^c which we denote by $1_{A_{N+1}^c} d(Y_K, X_{N+1}) \uparrow 1_{A_{N+1}^c}\infty$.[1] The triangle inequality of d implies that $d(Y_K, X_{N+1}) \leq d(Y_K, X_N) + d(X_N, X_{N+1})$. As $d(X_N, X_{N+1}) \in L^0$, it holds that $1_{A_{N+1}^c} d(Y_K, X_N) \uparrow 1_{A_{N+1}^c}\infty$. Since $(Y_K)_{K\in\mathbb{N}(\mathcal{F})} \subseteq L(X_{N+1}) \subseteq L(X_N)$, it follows that $A_{N+1}^c \subseteq A_N^c$. Further, it holds that $\wedge_{N\in\mathbb{N}(\mathcal{F})} A_N^c = \emptyset$. Indeed suppose there is some $A^c \in \mathcal{F}$, $P(A^c) > 0$ with $\wedge_{N\in\mathbb{N}(\mathcal{F})} A_N^c = A^c$. Then, there exists a decreasing local sequence $(X_N)_{N\in\mathbb{N}(\mathcal{F})}$ such that $1_{A^c} d(X_{N+1}, X_N) \geq 1_{A^c}$ for every $N \in \mathbb{N}(\mathcal{F})$, contradicting regularity. Therefore, we conclude $\vee_{N\in\mathbb{N}(\mathcal{F})} A_N = \Omega$. There exists a partition $(C_k)_{k\in\mathbb{N}}$ such that for every $k \in \mathbb{N}$ it holds $C_k \subseteq A_{N_k}$ for some $N_k \in \mathbb{N}(\mathcal{F})$.[2] Hence, it follows that $A_N^c = \emptyset$ for all $N \in \mathbb{N}(\mathcal{F})$, $N \geq N_0$, where $N_0 := \sum_{k\in\mathbb{N}} 1_{C_k} N_k$. Indeed, it holds $A_{N_0}^c = \cup_{k\in\mathbb{N}}(C_k \cap A_{N_k}^c) = \emptyset$ by construction. In that way, we found $N_0 \in \mathbb{N}(\mathcal{F})$ such that $d(X_{N+1}, X_N) \geq \operatorname{ess\,sup}_{X\in L(X_N)} d(X, X_N) - 1/N$ for all $N \in \mathbb{N}(\mathcal{F})$, $N \geq N_0$.

Thus, it holds that

$$
\operatorname*{ess\,sup}_{X\in L(X_N)} d(X, X_N) \leq d(X_{N+1}, X_N) + \frac{1}{N},
$$

[1] For example choose Y_K such that $1_{A_{N+1}^c} d(Y_K, X_{N+1}) \geq 1_{A_{N+1}^c} K$.

[2] This property is called countable chain condition and is always fulfilled by a σ-algebra of a measure space.

for all $N \in \mathbb{N}(\mathcal{F})$, $N \geq N_0$. In particular, since $X_M \in L(X_N)$ for all $M \geq N$, it follows that
$$d(X_M, X_N) \leq d(X_{N+1}, X_N) + \frac{1}{N}.$$
Regularity implies $d(X_{N+1}, X_N) + 1/N \to 0$ causing $(X_N)_{N\in\mathbb{N}(\mathcal{F})}$ to be a decreasing Cauchy sequence which has to converge to some $\bar{X} \in \mathcal{X}$, as $\mathcal{X}$ is $\preceq$-complete. By lower closedness it follows that $\bar{X} \preceq X_N$ for all $N \in \mathbb{N}(\mathcal{F})$, in particular $\bar{X} \preceq Y$.

For the second part of the proof of (I1), consider $X' \in L(\bar{X})$. By transitivity $X' \preceq \bar{X} \preceq X_N$ and thereby $X' \in L(X_N)$ for all $N \in \mathbb{N}(\mathcal{F})$. This implies

$$d(X', X_N) \leq \operatorname*{ess\,sup}_{X\in L(X_N)} d(X, X_N) \leq d(X_{N+1}, X_N) + \frac{1}{N}, \ \forall N \geq N_0.$$

Consequently, $X_N \to X'$, hence $X' = \bar{X}$ and therefore $L(\bar{X}) = \{\bar{X}\}$. This completes the proof.

Proof of (I2): Pick $\bar{X}$ satisfying the conclusions of (I1). If $T(X) \cap S(X)$ is nonempty for every $X \in \mathcal{X}$, then in particular $T(\bar{X}) \cap L(\bar{X}) \neq \emptyset$. Since $L(\bar{X}) = \{\bar{X}\}$ by (I1), $\bar{X}$ has to be a fixed point of T. Moreover, $T(X) \subseteq S(X)$, for every $X \in \mathcal{X}$, implies $T(\bar{X}) \subseteq L(\bar{X})$. Due to (I1), it holds that $L(\bar{X}) = \{\bar{X}\}$ and hence $\bar{X}$ is an invariant point of T in that case.

Proof of (I3): Suppose for every $X \in L(X_0)\backslash\mathcal{M}$ there exist $X' \in L(X)\backslash\{X\}$. By (I1), there exists $\bar{X} \in L(X_0)$ such that $L(\bar{X}) = \{\bar{X}\}$. Since $L(\bar{X})\backslash\{\bar{X}\} = \emptyset$, $\bar{X} \in \mathcal{M}$ has to hold and hence $\bar{X} \in L(X_0) \cap \mathcal{M}$. $\square$

Remark 2.11. Conversely, (I1) can be proven using (I2). To see this, replace $\mathcal{X}$ in Theorem (I2) by $L(X_0)$, for an arbitrary $X_0 \in \mathcal{X}$, and consider the map $T(X) := L(X)$ which of course satisfies $T(X) \cap L(X) \neq \emptyset$. Moreover, (I1) can be proven using (I3). Indeed, let the assumptions of (I1) be in force. Define $\mathcal{M} := \{X \in \mathcal{X} : L(X) = \{X\}\}$ which is a σ-stable set. If $X \notin \mathcal{M}$, then there exists $X' \in \mathcal{X}$ such that $X' \neq X$, $X' \preceq X$ and hence the assumption of (I3) is satisfied. By Theorem (I3), there exists $\bar{X} \in L(X_0) \cap \mathcal{M}$, in particular $L(\bar{X}) = \{\bar{X}\}$.

We can apply (I1) of the previous theorem as follows. Fix $Z \in \mathcal{X}$, $A \in \mathcal{F}$ and consider $Y = 1_A Y + 1_{A^c} Z$. We only want to vary the part on A. Applying the previous theorem yields $\bar{X}$ with $\bar{X} \preceq Y$ and $L(\bar{X}) = \{\bar{X}\}$. Hence, $\bar{X} \preceq 1_A Y + 1_{A^c} Z$ and thereby $1_A \bar{X} + 1_{A^c} Z \preceq 1_A Y + 1_{A^c} Z$. Moreover, $1_A X' + 1_{A^c} \bar{X} \preceq \bar{X}$ yields $1_A X' + 1_{A^c} \bar{X} = \bar{X}$, implying in turn $1_A X' + 1_{A^c} Z = 1_A \bar{X} + 1_{A^c} Z$. In this manner, we obtain results for elements differing on some set A only, which is one purpose of conditional theory.

Definition 2.12. Let $(\mathcal{X}, d)$ be an L^0-metric module, $\mathcal{Y}$ a nonempty σ-stable subset of some L^0-module and $\mathcal{M} \in S(\mathcal{X} \times \mathcal{Y})$ such that $\mathcal{M}(X) := \{(X', Y) \in \mathcal{M} : X' = X\}$ is nonempty for every $X \in \mathcal{X}$. Given a local preorder $\preceq$ on $\mathcal{M}$, we define $X' \preccurlyeq X$ if and only if

$$\forall Y, (X,Y) \in \mathcal{M},\ \exists Y', (X',Y') \in \mathcal{M} :\ (X',Y') \preceq (X,Y)$$

and $X' \curlyeqprec X$ if and only if

$$\forall Y', (X',Y') \in \mathcal{M},\ \exists Y, (X,Y) \in \mathcal{M} :\ (X',Y') \preceq (X,Y).$$

Further, we denote $\mathcal{M}_{\mathcal{Y}}(X) := \{Y \in \mathcal{Y} :\ (X,Y) \in \mathcal{M}\}$.

The set $\mathcal{M}(X)$ belongs to $S(\mathcal{X} \times \mathcal{Y})$, $X \in \mathcal{X}$. Indeed, given a sequence $(X, Y_n)_{n\in\mathbb{N}}$ in $\mathcal{M}(X)$ and a partition $(A_n)_{n\in\mathbb{N}}$, it holds that $\sum_{n\in\mathbb{N}} 1_{A_n}(X,Y_n) = \left(X, \sum_{n\in\mathbb{N}} 1_{A_n} Y_n\right)$ is in $\mathcal{M}(X)$, since $\mathcal{M}$ is σ-stable. Moreover, $\mathcal{M}\left(\sum_{n\in\mathbb{N}} 1_{A_n} X_n\right) = \sum_{n\in\mathbb{N}} 1_{A_n} \mathcal{M}(X_n)$, where one inclusion follows by σ-stability of $\mathcal{Y}$ and the other by σ-stability of $\mathcal{M}$. The set $\mathcal{M}_{\mathcal{Y}}(X)$ is an element of $S(\mathcal{X})$. Indeed, consider a sequence $(Y_n)_{n\in\mathbb{N}}$ in $\mathcal{M}_{\mathcal{Y}}(X)$ and a partition $(A_n)_{n\in\mathbb{N}}$. It holds that $\left(X, \sum_{n\in\mathbb{N}} 1_{A_n} Y_n\right) = \sum_{n\in\mathbb{N}} 1_{A_n}(X,Y_n) \in \mathcal{M}$, as $\mathcal{M}$ is σ-stable.

Both $\preccurlyeq$ and $\curlyeqprec$ are local preorders on $\mathcal{X}$. Indeed, transitivity and reflexivity follow immediately. To show the locality, consider a partition $(A_n)_{n\in\mathbb{N}}$ and sequences $(X_n)_{n\in\mathbb{N}}$ and $(X'_n)_{n\in\mathbb{N}}$ such that $X'_n \preccurlyeq X_n$ for all $n \in \mathbb{N}$. Therefore, for every $n \in \mathbb{N}$ it holds that for all $Y_n \in \mathcal{M}_{\mathcal{Y}}(X_n)$ there exists $Y'_n \in \mathcal{M}_{\mathcal{Y}}(X'_n)$ such that $(X'_n, Y'_n) \preceq (X_n, Y_n)$. To verify $\sum_{n\in\mathbb{N}} 1_{A_n} X' \preccurlyeq \sum_{n\in\mathbb{N}} 1_{A_n} X$, we have to consider $Y = \sum_{n\in\mathbb{N}} 1_{A_n} Y_n$ in $\mathcal{M}\left(\sum_{n\in\mathbb{N}} 1_{A_n} X\right)$. By the above characterization, we define the corresponding $Y' = \sum_{n\in\mathbb{N}} 1_{A_n} Y'_n$ which is in $\mathcal{M}\left(\sum_{n\in\mathbb{N}} 1_{A_n} X'\right)$. Due to locality of $\preceq$, it follows $\left(\sum_{n\in\mathbb{N}} 1_{A_n} X', Y'\right) \preceq \left(\sum_{n\in\mathbb{N}} 1_{A_n} X, Y\right)$ yielding the claim.

Theorem 2.13. *Let the following assumptions be satisfied:*

(M1) The pair $(\mathcal{X}, d)$ is an L^0-metric module, $\mathcal{Y}$ a nonempty σ-stable subset of some L^0-module, $\mathcal{M}$ in $S(\mathcal{X} \times \mathcal{Y})$ and $\mathcal{M}(X) \neq \emptyset$ for all $X \in \mathcal{X}$.

(M2) The relation $\preceq$ is a local preorder on $\mathcal{X} \times \mathcal{Y}$.

(M3) If $((X_N, Y_N))_{N\in\mathbb{N}(\mathcal{F})} \subseteq \mathcal{M}$ is a decreasing local sequence, that is

$$\forall N \in \mathbb{N}(\mathcal{F}) :\ (X_{N+1}, Y_{N+1}) \preceq (X_N, Y_N)$$

and $(X_N)_{N\in\mathbb{N}(\mathcal{F})}$ converges to $X \in \mathcal{X}$, then there exists $Y \in \mathcal{Y}$ such that $(X,Y) \in \mathcal{M}$ and

$$\forall N \in \mathbb{N}(\mathcal{F}) :\ (X,Y) \preceq (X_N, Y_N).$$

(M4) If $((X_N, Y_N))_{N\in\mathbb{N}(\mathcal{F})} \subseteq \mathcal{M}$ is a decreasing local sequence, then defining $Z_N := d(X_N, X_{N+1})$ it holds that $Z_N \to 0$.

Then, for each $X_0 \in \mathcal{X}$ there exists $\bar{X} \in \mathcal{X}$ such that *(i)* $\bar{X} \preccurlyeq X_0$.

(ii) *If* $X \preccurlyeq \bar{X}$, *then* $X = \bar{X}$.

Proof. We want to apply Theorem 2.10 to the local preorder $\preccurlyeq$. Concerning regularity, consider a local sequence $(X_N)_{N\in\mathbb{N}(\mathcal{F})}$ which is decreasing with respect to $\preccurlyeq$. In particular, the sequence $(X_n)_{n\in\mathbb{N}}$ fulfills

$$\forall Y_n \in \mathcal{M}_{\mathcal{Y}}(X_n)\ \exists Y_{n+1} \in \mathcal{M}_{\mathcal{Y}}(X_{n+1}):\ (X_{n+1}, Y_{n+1}) \preceq (X_n, Y_n). \tag{2.2}$$

For $Y_0 \in \mathcal{M}_{\mathcal{Y}}(X_0)$, we pick $Y_1 \in \mathcal{M}_{\mathcal{Y}}(X_1)$ via (2.2) such that $(X_1, Y_1) \preceq (X_0, Y_0)$. We choose $Y_2 \in \mathcal{M}_{\mathcal{Y}}(X_2)$ via (2.2) such that $(X_2, Y_2) \preceq (X_1, Y_1)$. Following this procedure, we obtain a sequence $(Y_n)_{n\in\mathbb{N}}$ and consider the corresponding local sequence $(Y_N)_{N\in\mathbb{N}(\mathcal{F})}$. By locality, $((X_N, Y_N))_{N\in\mathbb{N}(\mathcal{F})} \subseteq \mathcal{M}$ is decreasing with respect to $\preceq$. By (M4), we obtain $d(X_{N+1}, X_N) \to 0$ as desired.

As a next step, we show that $\preccurlyeq$ is lower closed. Consider a local sequence $(X_N)_{N\in\mathbb{N}(\mathcal{F})}$ decreasing with respect to $\preccurlyeq$ and converging to $X \in \mathcal{X}$. We have to show that $X \preccurlyeq X_N$ for each $N \in \mathbb{N}(\mathcal{F})$. Fix $N_0 \in \mathbb{N}(\mathcal{F})$. By (2.2) and the subsequent construction, we find $Y_{N_0+1} \in \mathcal{M}_{\mathcal{Y}}(X_{N_0+1})$ such that $(X_{N_0+1}, Y_{N_0+1}) \preceq (X_{N_0}, Y_{N_0})$ and, as before, a local sequence $((X_{N_0+M}, Y_{N_0+M}))_{M\in\mathbb{N}(\mathcal{F})}$ decreasing with respect to $\preceq$. It still holds that $X_{N_0+M} \to X$. Assumption (M3) implies the existence of $Y \in \mathcal{M}_{\mathcal{Y}}(X)$ such that for each $M \in \mathbb{N}(\mathcal{F})$

$$(X, Y) \preceq (X_{N_0+M}, Y_{N_0+M}) \preceq (X_{N_0}, Y_{N_0}).$$

This procedure is applicable for every $N_0 \in \mathbb{N}(\mathcal{F})$ (the corresponding $Y \in \mathcal{M}_{\mathcal{Y}}(X)$ may depend on N_0). Hence, the lower closedness of $\preccurlyeq$ is proven.

The final step of the proof is an application of Theorem 2.10 to the L^0-metric module $(\mathcal{X}, d)$ and the relation $\preccurlyeq$ in order to obtain *(i)* and *(ii)*. □

Analyzing the proof above, note that it is not possible to show the regularity and lower closedness if $\preccurlyeq$ is simply replaced by $\curlyeqprec$. Hence, we have to modify the proof to obtain the result for $\curlyeqprec$.

Theorem 2.14. *Let the following assumptions be satisfied:*

(M1) and (M2) as in Theorem 2.13.

(M3') If $((X_N, Y_N))_{N\in\mathbb{N}(\mathcal{F})} \subseteq \mathcal{M}$ *is an increasing sequence, that is*

$$\forall N \in \mathbb{N}(\mathcal{F}):\ (X_N, Y_N) \preceq (X_{N+1}, Y_{N+1})$$

and $(X_N)_{N\in\mathbb{N}(\mathcal{F})}$ *converges to* $X \in \mathcal{X}$, *then there exists* $Y \in \mathcal{Y}$ *such that* $(X, Y) \in \mathcal{M}$ *and*

$$\forall N \in \mathbb{N}(\mathcal{F}):\ (X_N, Y_N) \preceq (X, Y).$$

(M4') *If* $((X_N, Y_N))_{N\in\mathbb{N}(\mathcal{F})} \subseteq \mathcal{M}$ *is an increasing local sequence, then it holds that* $\lim_{N\to\infty} d(X_{N+1}, X_N) = 0$.

Then, for each $X_0 \in \mathcal{X}$ *there exists* $\bar{X} \in \mathcal{X}$ *such that the following is fulfilled:*

(i) $X_0 \preccurlyeq \bar{X}$.

(ii) *If* $\bar{X} \preccurlyeq X$*, then* $X = \bar{X}$.

Proof. Define the binary relation $\preceq'$ on $\mathcal{X} \times \mathcal{Y}$ by

$$(X', Y') \preceq' (X, Y) \quad \Longleftrightarrow \quad (X, Y) \preceq (X', Y').$$

It holds that $\preceq'$ is a local preorder. Moreover, a sequence $((X_N, Y_N))_{N\in\mathbb{N}(\mathcal{F})} \subseteq \mathcal{X} \times \mathcal{Y}$ is decreasing with respect to $\preceq'$ if and only if it is increasing with respect to $\preceq$. Hence (M3') and (M4') of Theorem 2.14 are satisfied for $\preceq'$ if and only if (M3) and (M4) of Theorem 2.13 are satisfied for $\preceq$. We apply Theorem 2.13 to obtain $\bar{X} \in \mathcal{X}$ such that $\mathcal{M}_{\mathcal{Y}}(\bar{X}) \neq \emptyset$ and the following is fulfilled:

(i') $\bar{X} \preccurlyeq' X_0$.

(ii') If $X \preccurlyeq' \bar{X}$, then $X = \bar{X}$.

Observing that $X' \preccurlyeq' X$ if and only if $X \preccurlyeq X'$ we note that (i') and (ii') are equivalent to (i) and (ii), respectively. This completes the proof. $\square$

Remark 2.15. In the case of $\mathcal{M}(X)$ consisting of only a single element, that is $\mathcal{M}(X) = \{(X, Y)\}$, the set $\mathcal{M} \in S(\mathcal{X} \times \mathcal{Y})$ defines a local function $f : \mathcal{X} \to \mathcal{Y}$. The local relation $\preceq$ coincides with $\preccurlyeq$, and they compare arguments and values of f at the same time:

$$X' \preccurlyeq X \quad \Longleftrightarrow \quad (X', f(X')) \preceq (X, f(X)).$$

Remark 2.16. In the deterministic setting, that is if $\mathcal{F}$ is the trivial σ-algebra, Assumption (M3) or (M3') of Theorem 2.13 or 2.14, respectively, coincides with Assumption (H1) in [55] if $\mathcal{Y}$ is assumed to be a topological linear space. To verify Assumption (M3) or (M3') in specific cases might be difficult, compare the discussion in [55] and [56], Section 3.10. Moreover, note that Assumption (2) of Theorem 1 of Brézis and Browder [16] has a similar structure, does however not deal with set products.

2.3 Ekeland's Variational Principle

Throughout this section, let $\mathcal{X}$ be a nonempty, σ-stable L^0-module.

2.3.1 Local Premetric and Uniform Structure

Definition 2.17. A σ-stable subset $\mathcal{Y}$ of an L^0-module is a local monoid if there exists a local function $\oplus : \mathcal{Y} \times \mathcal{Y} \to \mathcal{Y}$ which is associative and has a neutral element θ. If $X \oplus Y = Y \oplus X$ for every $X, Y \in \mathcal{Y}$, we call $\mathcal{Y}$ commutative. A triple $(\mathcal{Y}, \oplus, \leq)$ is called local preordered monoid if $(\mathcal{Y}, \oplus)$ is a commutative local monoid, $\leq$ is a local preorder and it holds that

$$\text{(Add)} \quad X \leq Y \quad \Longrightarrow \quad X \oplus Z \leq Y \oplus Z, \ \forall Z \in \mathcal{Y}.$$

If further $\leq$ is antisymmetric, we call $(\mathcal{Y}, \oplus, \leq)$ local ordered monoid.

For instance L^0 is a local monoid for the addition and $\theta = 0$. Using the P-almost greater/equal order makes L^0 to be a local ordered monoid. The set $\mathcal{Y} = [1, \infty) := \{X \in L^0 : X \geq 1\}$ together with the multiplication operation and the P-almost greater/equal order is a local preordered monoid. However, $[1, \infty)$ is not a group since there is no inverse for $3 \cdot 1_\Omega$.

Definition 2.18. Let $(Y, \oplus, \leq)$ be a local preordered monoid with neutral element $\theta \in Y$. A local function $\Phi : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ is called a local order premetric if the following conditions are satisfied:

(P1) $\forall X \in \mathcal{X}$: $\theta = \Phi(X, X)$;

(P2) $\forall X_1, X_2 \in \mathcal{X}$: $\theta \leq \Phi(X_1, X_2)$;

(P3) $\forall X_1, X_2, X_3 \in \mathcal{X}$: $\Phi(X_1, X_3) \leq \Phi(X_1, X_2) \oplus \Phi(X_2, X_3)$.

Any L^0-metric is a local order premetric. Note that no kind of symmetry is required in the results below.

Remark 2.19. A priori the local order on Y does not have to be complete. However, a local order premetric mapping to $\mathcal{Y}$ has to fulfill (P2), showing that $Y \geq \theta$ needs to be fulfilled for every $Y \in \mathcal{Y}$ with $Y = \Phi(.,.)$. Hence, instead of $(\mathcal{Y}, \leq, \oplus)$ we actually consider $(\Phi(\mathcal{X} \times \mathcal{X}), \leq, \oplus)$ which is a local preordered monoid as well but with slightly more structure.

We denote $\Delta = \{(X, X) : X \in \mathcal{X}\}$.

Definition 2.20. A local uniform structure is a nonempty family $\mathscr{U}$ in $S(S(\mathcal{X} \times \mathcal{X}))$ such that the following is fulfilled:

(U1) $\Delta \subseteq \mathcal{U}$ for all $\mathcal{U} \in \mathscr{U}$;

(U2) $\mathcal{U} \in \mathscr{U}$ and $\mathcal{U} \subseteq \mathcal{V}$ for $\mathcal{V} \in S(\mathcal{X} \times \mathcal{X})$ implies $\mathcal{V} \in \mathscr{U}$;

(U3) $\mathcal{U} \cap \mathcal{V} \in \mathcal{U}$ if $\mathcal{U}, \mathcal{V} \in \mathscr{U}$;

(U4) For every $\mathcal{U} \in \mathscr{U}$ there exists $\mathcal{V} \in \mathscr{U}$ such that $(X,Y) \in \mathcal{V}$, $(Y,Z) \in \mathcal{V}$ always implies $(X,Z) \in \mathcal{U}$.

(U5) If $\mathcal{U} \in \mathscr{U}$, then also $\{(Y,X) : (X,Y) \in \mathcal{U}\} \in \mathscr{U}$.

A local uniform structure is defined analogously to conditional neighborhood bases (compare Definition 4.17) or conditional filters (compare [31]).

Definition 2.21. Let $\mathscr{U}$ be a local uniform structure. Then $\mathscr{V} \in S(S(\mathcal{X} \times \mathcal{X}))$ is called local fundamental system of entourages of $\mathscr{U}$ if for every $\mathcal{U} \in \mathscr{U}$ there is a $\mathcal{V} \in \mathscr{V}$ such that $\mathcal{V} \subseteq \mathcal{U}$, or equivalently

$$\mathscr{U} = \{\mathcal{U} \in S(\mathcal{X} \times \mathcal{X}) : \exists \mathcal{V} \in \mathscr{V}, \mathcal{V} \subseteq \mathcal{U}\}.$$

In the following lemma, recall that $Y > 0$ in L^0 expresses $Y \geq 0$ and that there exists no $A \in \mathcal{F}_+$ with $1_A Y = 0$.

Lemma 2.22. *Let $(\mathcal{X}, d)$ be an L^0-metric module. Then, the collection $(\mathcal{U}_Y)_{Y \in L^0, Y>0}$ with*

$$\mathcal{U}_Y = \{(X_1, X_2) \in \mathcal{X} \times \mathcal{X} : d(X_1, X_2) \leq Y\},$$

forms a local fundamental system of a uniform structure.

Proof. The fact that $\mathcal{U}_Y$ is in $S(\mathcal{X} \times \mathcal{X})$ is due to the property that d and $\leq$ are local functions. We show that $(\mathcal{U}_Y)_{Y \in \mathcal{Y}, Y>0}$ is σ-stable. To this end, consider sets $(\mathcal{U}_{Y_n})_{n\in\mathbb{N}}$ and a partition $(A_n)_{n\in\mathbb{N}}$. It holds $\sum_{n\in\mathbb{N}} 1_{A_n} Y_n > 0$ and $\sum_{n\in\mathbb{N}} 1_{A_n} \mathcal{U}_{Y_n} = \mathcal{U}_{\sum_{n\in\mathbb{N}} 1_{A_n} Y_n}$. Indeed, consider an element $(X^1, X^2) \in \sum_{n\in\mathbb{N}} 1_{A_n} \mathcal{U}_{Y_n}$ meaning $X^1 = \sum_{n\in\mathbb{N}} 1_{A_n} X_n^1$, $X^2 = \sum_{n\in\mathbb{N}} 1_{A_n} X_n^2$ with $d(X_n^1, X_n^2) \leq Y_n$. Since d is local, it follows $d(X^1, X^2) = \sum_{n\in\mathbb{N}} 1_{A_n} d(X_n^1, X_n^2) \leq \sum_{n\in\mathbb{N}} 1_{A_n} Y_n$, which yields the claim. Reversely, consider $(X^1, X^2) \in \mathcal{U}_{\sum 1_{A_n} Y_n}$, thereby $d(X^1, X^2) \leq \sum_{n\in\mathbb{N}} 1_{A_n} Y_n$ and hence $1_{A_n} d(X^1, X^2) + 1_{A_n^c} Y \leq 1_{A_n} Y_n + 1_{A_n^c} Y$ for all $n \in \mathbb{N}$ and arbitrary $Y \in \mathcal{Y}$. Thus, defining $X_n^1 = 1_{A_n} X^1 + 1_{A_n^c} X$ and $X_n^2 = 1_{A_n} X^2 + 1_{A_n^c} X$ for some arbitrary $X \in \mathcal{X}$ yields $d(X_n^1, X_n^2) \leq Y_n$. Doing so for every n yields the claim, since (X^1, X^2) can be written as $\sum_{n\in\mathbb{N}} 1_{A_n} (X_n^1, X_n^2)$, which is the combination of elements in $\mathcal{U}_{Y_n}$ we aimed at. The fact that $\sum_{n\in\mathbb{N}} 1_{A_n} Y_n > 0$ follows from the locality of $\leq$.

To show that $(\mathcal{U}_Y)_{Y \in L^0, Y>0}$ is a local fundamental system of entourages, we first easily note that Δ is contained in every $\mathcal{U}_Y$ as $d(X,X) = 0 < Y$, since we index by $Y > 0$ only. Property (U2) is fulfilled by construction and (U5) follows by symmetry of d. To show Property (U3), let $\mathcal{U}_{Y_1}, \mathcal{U}_{Y_2} \in \mathscr{U}$. By defining $\bar{Y} = Y_1 \wedge Y_2$ where $\wedge$ denotes the essential infimum in L^0, it holds that $\mathcal{U}_{\bar{Y}} \subseteq \mathcal{U}_{Y_1} \cap \mathcal{U}_{Y_2}$. To prove Property (U4), consider $\mathcal{U}_Y \in \mathscr{U}$. Defining $\mathcal{V} = \mathcal{U}_{Y/2}$ yields the property demanded in (U4). Indeed, let $(X,Y), (Y,Z) \in \mathcal{V}$, thereby $d(X,Y) < Y/2$ and $d(Y,Z) \leq Y/2$. The triangle inequality of d implies $\phi(X,Z) \leq \phi(X,Y) \oplus \phi(Y,Z) = Y/2 + Y/2 = Y$, showing that $(X,Z) \in \mathcal{U}_Y$. □

A concept needed in the following is the sum using coefficients in $\mathbb{N}(\mathcal{F})$. The object $\sum_{K=0}^{N}$ has to be understood as follows. The expression N determines the partition $(A_n)_{n\in\mathbb{N}}$ via $A_n = \{N = n\}$ for every $n \in \mathbb{N}$ and on A_n we count from 1 up to n. Since we will only sum local functions f, we define $\sum_{K=0}^{N} f(X_K) := \sum_{n\in\mathbb{N}} \left(1_{\{N=n\}} \sum_{k=1}^{n} f(X_n)\right)$. The sum $\sum_{k=1}^{n}$ has to be understood with respect to $\oplus$

Definition 2.23. Let $\mathscr{U}$ be a local uniform structure on $\mathcal{X}$ and $(\mathcal{Y}, \oplus, \leq)$ a local preordered monoid with neutral element $\theta \in \mathcal{Y}$. A local order premetric Φ is called regular with respect to $Y_1, Y_2 \in \mathcal{Y}$ if it satisfies:

(P5) If $(X_N)_{N\in\mathbb{N}(\mathcal{F})} \subseteq \mathcal{X}$ is a local sequence and

$$\forall N \in \mathbb{N}(\mathcal{F}):\ Y_1 \oplus \sum_{K=0}^{N} \Phi(X_{K+1}, X_K) \leq Y_2,$$

then $(X_N)_{N\in\mathbb{N}(\mathcal{F})}$ is asymptotic, that is

$$\forall \mathcal{E} \in \mathscr{U}\ \exists N_{\mathcal{E}} \in \mathbb{N}(\mathcal{F})\ \forall N \geq N_{\mathcal{E}}:\ (X_{N+1}, X_N) \in \mathcal{E}.$$

Note that if $(\mathcal{X}, d)$ is an L^0-metric module we always consider the local fundamental system which is generated by all sets $\{(X,Y) \in \mathcal{X}\times\mathcal{X} : d(X,Y) \leq \varepsilon\}$, $\varepsilon \in L^0_{++}$, as in Lemma 2.22.

Lemma 2.24. *Let $(\mathcal{X}, d)$ be an L^0-metric module. Then, a local sequence $(X_N)_{N\in\mathbb{N}(\mathcal{F})}$ in $\mathcal{X}$ is asymptotic if and only if $\lim_{N\to\infty} d(X_{N+1}, X_N) = 0$.*

Proof. Recall, that $\lim_{N\to\infty} d(X_{N+1}, X_N) = 0$ expresses: For every $\varepsilon \in L^0_{++}$, there exists $N_0 \in \mathbb{N}(\mathcal{F})$ such that for all $N \geq N_0$ it holds $d(X_{N+1}, X_N) \leq \varepsilon$. Hence, by definition $\lim_{N\to\infty} d(X_{N+1}, X_N) = 0$ implies that the local sequence is asymptotic.

Reversely, suppose the local sequence is asymptotic. Defining

$$A = \vee\{B : 1_B d(X_{N+1}, X_N) \text{ converges to } 0\},$$

it holds that $1_A d(X_{N+1}, X_N)$ converges to zero. It holds that $1_C d(X_{N+1}, X_N) \not\to 0$ for all $C \subseteq A^c$, $C \neq \emptyset$. Hence, there exists some $\varepsilon \in L^0_{++}$ such that $d(X_{N+1}, X_N) > \varepsilon$ on A^c for sufficiently many N. Therefore, there does not exist $N_{\mathcal{E}}$ for $\mathcal{E} = \{(X,Y) \in \mathcal{X}\times\mathcal{X} : d(X,Y) \leq \varepsilon/2\}$ which is needed for property (P5). $\square$

Suppose, $\mathcal{Y}$ is not only a monoid but also a group, that is for $Y \neq \theta$ there exists an inverse with respect to $\oplus$. For $Y_1 = \theta$, the regularity assumption is trivially fulfilled. For $Y_1 \neq \theta$ it holds that $Y_1 \oplus \sum_{K=0}^{N} \Phi(X_{K+1}, X_K) \leq Y_2$ for all $N \in \mathbb{N}(\mathcal{F})$ if and only if $\theta \leq \sum_{K=0}^{N} \Phi(X_{K+1}, x_K) \leq Y_2 \oplus Y_1^{-1}$ for all $N \in \mathbb{N}(\mathcal{F})$. Hence, it is enough to assume that the boundedness from above of $\left\{\sum_{K=0}^{N} \Phi(X_{K+1}, X_K) :\ N \in \mathbb{N}(\mathcal{F})\right\}$ implies that $(X_N)_{N\in\mathbb{N}(\mathcal{F})}$ is asymptotic. This is still possible if $\mathcal{Y}$ is not a group but a local group, that is for every element Y for which there is no $A \in \mathcal{F}, P(A) > 0$ such that $1_A Y + 1_{A^c}\theta = \theta$, there exists an inverse. In that case, we can split the regularity inequality to $\{Y_1 = \theta\}$ and its complement and can argue separately by locality. An example of a local group is L^0 with the multiplication, since for every X with $P(X = 0) = 0$ there is an inverse, namely $1/X$.

2.3.2 Ekeland's Theorem

For the remainder of this chapter, $(\mathcal{X}, d)$ denotes an L^0-metric module, $(\mathcal{Y}, \oplus, \leq)$ a local preordered monoid and $\Phi : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ a local order premetric.

Theorem 2.25. *Let the following assumptions be satisfied:*

(A1) The local function $f : \mathcal{X} \to \mathcal{Y}$ and $\tilde{Y} \in \mathcal{Y}$ are such that

(i) $\tilde{Y} \leq f(X)$ for all $X \in \mathcal{X}$;

(ii) Φ is regular with respect to $\tilde{Y}$ and $f(X_0) \in \mathcal{Y}$ for $X_0 \in \mathcal{X}$;

(iii) If $(X_N)_{N\in\mathbb{N}(\mathcal{F})} \subseteq \mathcal{X}$ is a Cauchy sequence with

$$\forall N \in \mathbb{N}(\mathcal{F}) :\ f(X_{N+1}) \oplus \Phi(X_{N+1}, X_N) \leq f(X_N), \tag{2.3}$$

then it converges to some $X \in \mathcal{X}$.

(A2) If $(X_N)_{N\in\mathbb{N}(\mathcal{F})} \subseteq \mathcal{X}$ converges to $X \in \mathcal{X}$ and satisfies (2.3), then it holds that $f(X) \oplus \Phi(X, X_N) \leq f(X_N)$ for all $N \in \mathbb{N}(\mathcal{F})$.

Then, there exists $\bar{X} \in \mathcal{X}$ such that

(a) $f(\bar{X}) \oplus \Phi(\bar{X}, X_0) \leq f(X_0)$,

(b) if there is some $X \in \mathcal{X}, f(X) \oplus \Phi(X, \bar{X}) \leq f(\bar{X})$, then $X = \bar{X}$.

Proof. We prove the assertion by verifying the assumptions of Theorem 2.10 for the relation

$$X' \preceq X \quad :\Longleftrightarrow \quad f(X') \oplus \Phi(X', X) \leq f(X),$$

which then in turn yields the desired result

First, we show that $\preceq$ is a local relation. To this end, consider a partition $(A_n)_{n\in\mathbb{N}}$ and families $(X'_n)_{n\in\mathbb{N}}$ and $(X_n)_{n\in\mathbb{N}}$ such that $X'_n \preceq X_n$ for all $n \in \mathbb{N}$. Hence, it holds that $f(X'_n) \oplus \Phi(X'_n, X_n) \leq f(X_n)$. Denote $X = \sum_{n\in\mathbb{N}} 1_{A_n} X_n$ and $X' = \sum_{n\in\mathbb{N}} 1_{A_n} X'_n$. As $\oplus$ and $\leq$ are local operations, it follows that

$$f(X') \oplus \Phi(X', X) = \sum_{n\in\mathbb{N}} 1_{A_n}\big(f(X'_n) \oplus \Phi(X'_n, X_n)\big) \leq \sum_{n\in\mathbb{N}} 1_{A_n} f(X_n) = f(X).$$

The relation $\preceq$ is reflexive, since $\leq$ is reflexive and Φ satisfies (P1) of Definition 2.18. It is transitive due to (P3) of Definition 2.18, Property (Add) and the transitivity of $\leq$. Indeed, let $X \preceq Y \preceq Z$ that is $f(X) \oplus \Phi(X,Y) \leq f(Y)$ and $f(Y) \oplus \Phi(Y,Z) \leq f(Z)$. Then

$$\begin{aligned} f(X) \oplus \Phi(X,Z) &\leq f(X) \oplus (\Phi(X,Y) \oplus \Phi(Y,Z)) = (f(X) \oplus (\Phi(X,Y)) \oplus \Phi(Y,Z) \\ &\leq f(Y) \oplus \Phi(Y,Z) \leq f(Z) \end{aligned}$$

holds true, showing that $X \preceq Z$.

The $\preceq$–completeness of $\mathcal{X}$ follows directly from (A1) subitem (iii). The lower closedness is an immediate consequence of Assumption (A2). It remains to show that $\preceq$ is regular. To this end, let $(X_N)_{N\in\mathbb{N}(\mathcal{F})} \subseteq \mathcal{X}$ be such that $X_{N+1} \preceq X_N$ for all $N \in \mathbb{N}(\mathcal{F})$, that is

$$f(X_{N+1}) \oplus \Phi(X_{N+1}, X_N) \leq f(X_N).$$

Therefore, it holds that

$$f(X_{N+1}) \oplus \Phi(X_{N+1}, X_N) \oplus \Phi(X_N, X_{N-1}) \leq f(X_N) \oplus \Phi(X_N, X_{N-1}) \leq f(X_{N-1}).$$

Due to transitivity, it follows

$$f(X_{N+1}) \oplus \Phi(X_{N+1}, X_N) \oplus \Phi(X_N, X_{N-1}) \leq f(X_{N-1}).$$

Proceeding analogously, we obtain for each $N \in \mathbb{N}(\mathcal{F})$

$$f(X_{N+1}) \oplus \sum_{K=0}^{N} \Phi(X_{K+1}, X_K) \leq f(X_0).$$

Since $\tilde{Y} \leq f(X_M)$ for each $M \in \mathbb{N}(\mathcal{F})$, by (A1) subitem (i) it follows that

$$\tilde{Y} \oplus \sum_{K=0}^{N} \Phi(X_{K+1}, X_K) \leq f(X_0),$$

as a consequence of transitivity. Due to (A1) subitem (ii), Φ is regular with respect to $\tilde{Y}, f(X_0) \in \mathcal{Y}$, implying $d(X_{N+1}, X_N) \to 0$ and thereby showing regularity of $\preceq$. $\square$

Corollary 2.26. *Let $(\mathcal{X}, d)$ be a complete L^0-metric module and $f : \mathcal{X} \to \overline{L^0}$ a lower semicontinuous local function with* $\operatorname{ess\,inf} f \in L^0$*. Let further* $\operatorname{ess\,inf} f \leq f(X_0) \leq \operatorname{ess\,inf} f + \varepsilon$*, for $\varepsilon \in L^0_{++}$ and $\lambda \in L^0_{++}$. Then, there exists some $\bar{X}$ such that*

(i) $f(\bar{X}) \leq f(X_0)$*;*

(ii) $d(X_0, \bar{X}) \leq \lambda$*;*

(iii) for $X \in \mathcal{X}$ it holds that $f(X) + (\varepsilon/\lambda)d(X, \bar{X}) > f(\bar{X})$ on $\{X \neq \bar{X}\}$.

Proof. Define $\phi(X, Y) = (\varepsilon/\lambda)d(X, Y)$. Then by applying the previous theorem, to $Y = L^0$, we obtain $\bar{X} \in \mathcal{X}$ and a) and b) become:

(a) $f(\bar{X}) + (\varepsilon/\lambda)d(\bar{X}, X_0) \leq f(X_0)$,

(b) if there is some $X \in \mathcal{X}$ with $f(X) + (\varepsilon/\lambda)d(X, \bar{X}) \leq f(\bar{X})$, then $X = \bar{X}$.

Since d maps to L^0_+, *(a)* implies (i) and $d(\bar{X}, X_0) \leq (\lambda/\varepsilon)(f(X_0) - f(\bar{X}))$. As $f(\bar{X}) \leq f(X_0)$ and $f(X_0) \leq \inf f + \varepsilon$, it follows that $f(X_0) - f(\bar{X}) \leq \varepsilon$ which yields (ii). To obtain (iii) from b), consider an arbitrary X and define $A_X = \vee\{A : 1_A(f(X) + (\varepsilon/\lambda)d(X, \bar{X})) \leq 1_A f(\bar{X})\}$. Hence, it holds $f(X) + (\varepsilon/\lambda)d(X, \bar{X}) > f(\bar{X})$ on A^c_X. If we apply b) on $1_{A_X} X + 1_{A^c_X} \bar{X}$, it follows that $1_{A_X} X + 1_{A^c_X} \bar{X} = \bar{X}$. This means that $1_{A_X} X = 1_{A_X} \bar{X}$, thereby $A_X \subseteq \{X = \bar{X}\}$. The fact that $A_X \supseteq \{X = \bar{X}\}$ is obvious, so it holds $A_X = \{X = \bar{X}\}$ which yields the claim.

It remains to show that the previous theorem is applicable. The lower semicontinuity of f implies (A2) and the completeness of $(\mathcal{X}, d)$ implies $(A1)(iii)$. In $(A2)(i)$, we may choose $\bar{Y} = \operatorname{ess\,inf} f$. Hence, it remains to show that d is regular with respect to $\operatorname{ess\,inf} f, f(X_0)$, that is $(\operatorname{ess\,inf} f) \cdot \left(\sum_{K=1}^N d(X_{K+1}, X_K)\right) \leq f(X_0)$ for every $N \in \mathbb{N}(\mathcal{F})$ implies $\lim_{N\to\infty} d(X_{N+1}, X_N) = 0$. Indeed, assuming to the opposite that $(d(X_{N+1}, X_N))_{N\in\mathbb{N}(\mathcal{F})}$ does not converge to 0 yields some $N_0 \in \mathbb{N}(\mathcal{F})$, $A \in \mathcal{F}_+$ and $\mu \in L^0_{++}$ such that there exist sufficiently many k such that $d(X_{N_0+k+1}, X_{N_0+k}) > \mu$ on A. Consequently, $\sum_{K=1}^N d(X_{K+1}, X_K) > f(X_0)$ on A for sufficiently large N, a contradiction. □

Remark 2.27. Note that it is not necessary to assume $\mathcal{Y}$ to be a (local) group. Thus, Theorem 2.25 generalizes the result of [64] with respect to the image space $\mathcal{Y}$, even if we consider the trivial σ-algebra $\mathcal{F} = \{\emptyset, \Omega\}$.

To verify that Property (A2) of Theorem 2.25 is fulfilled can be delicate. Hence, for the deterministic setting in [64] a sufficient condition implying Property (A2) was given. We follow this idea, adapted to our setting, below.

Definition 2.28. A local function $f : \mathcal{X} \to \mathcal{Y}$ is called lower monotone if, for each local sequence $(X_N)_{N\in\mathbb{N}(\mathcal{F})} \subseteq \mathcal{X}$ converging to $X \in \mathcal{X}$ and satisfying $f(X_{N+1}) \leq f(X_N)$ for all $N \in \mathbb{N}(\mathcal{F})$, the inequality $f(X) \leq f(X_N)$ holds true for all $N \in \mathbb{N}(\mathcal{F})$. A local order premetric $\Phi : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ is called lower monotone with respect to the first variable if, for each $X' \in \mathcal{X}$ and each local sequence $(X_N)_{N\in\mathbb{N}(\mathcal{F})} \subseteq \mathcal{X}$ converging to $X \in \mathcal{X}$ and $Y_1, Y_2 \in \mathcal{Y}$, the condition

$$\forall N \in \mathbb{N}(\mathcal{F}) : \; Y_1 \oplus \Phi(X_N, X') \leq Y_2$$

implies $Y_1 \oplus \Phi(X, X') \leq Y_2$.

Lemma 2.29. *Let $(\mathcal{X}, d)$ be an L^0-metric module and $(\mathcal{Y}, \oplus, \leq)$ a local ordered monoid. Let the local function $f : \mathcal{X} \to \mathcal{Y}$ be lower monotone and the local order premetric $\Phi : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ be lower monotone with respect to the first variable. Then, (A2) of Theorem 2.25 is satisfied.*

Proof. Consider a local sequence $(X_N)_{N\in\mathbb{N}(\mathcal{F})} \subseteq \mathcal{X}$ converging to $X \in \mathcal{X}$ such that

$$\forall N \in \mathbb{N}(\mathcal{F}) : \; f(X_{N+1}) \oplus \Phi(X_{N+1}, X_N) \leq f(X_N). \tag{2.4}$$

Since $\theta \leq \Phi(X_{N+1}, X_N)$, it follows that

$$f(X_{N+1}) = f(X_{N+1}) \oplus \theta \leq f(X_{N+1}) \oplus \Phi(X_{N+1}, X_N) \leq f(X_N),$$

which yields $f(X_{N+1}) \leq f(X_N)$ for all $N \in \mathbb{N}(\mathcal{F})$, because $\leq$ is transitive. The lower monotonicity of f implies $f(X) \leq f(X_N)$ for all $N \in \mathbb{N}(\mathcal{F})$.

Applying (2.4) twice, we obtain

$$\begin{aligned} f(X_N) &\geq f(X_{N+1}) \oplus \Phi(X_{N+1}, X_N) \geq (f(X_{N+2}) \oplus \Phi(X_{N+2}, X_{N+1})) \oplus \Phi(X_{N+1}, X_N) \\ &= f(X_{N+2}) \oplus (\Phi(X_{N+2}, X_{N+1}) \oplus \Phi(X_{N+1}, X_N)) \geq f(X_{N+2}) \oplus \Phi(X_{N+2}, X_N), \end{aligned}$$

where we used the triangle inequality for Φ and Property (Add) of $\oplus$. By induction, we deduce that $f(X_N) \geq f(X_{N+k}) \oplus \Phi(X_{N+k}, X_N)$ for any $N \in \mathbb{N}(\mathcal{F})$, $k \in \mathbb{N}$. Due to locality of $\oplus, \leq, f$ and Φ, it also follows that $f(X_N) \geq f(X_{\bar{N}}) \oplus \Phi(X_{\bar{N}}, X_N)$ for any $\bar{N} \geq N$. Hence, fixing $N_0 \in \mathbb{N}(\mathcal{F})$, for $N \geq N_0$, we obtain that

$$f(X) \oplus \Phi(X_N, X_{N_0}) \leq f(X_N) \oplus \Phi(X_N, X_{N_0}) \leq f(X_{N_0}).$$

Define the local sequence $(Z_N)_{N\in\mathbb{N}(\mathcal{F})}$ by $Z_N := 1_{\{N \geq N_0\}} X_N + 1_{\{N < N_0\}} X_{N_0}$. Since also $Z_N \to X$, it follows that

$$f(X) \oplus \Phi(Z_N, Z_{N_0}) \leq f(Z_{N_0}),$$

for all $N \in \mathbb{N}(\mathcal{F})$. Thus, the lower monotonicity of Φ implies

$$f(X) \oplus \Phi(X, X_{N_0}) \leq f(X_{N_0}),$$

where we used $Z_{N_0} = X_{N_0}$. This holds for any N_0 which is what we aimed at. □

Theorem 2.25 applies to local functions, so in particular to local set-valued functions to obtain Ekeland type theorems for local set-valued maps. As a preparation, we show how to transfer the concepts from elements to sets.

Definition 2.30. Let $(\mathcal{Y}, \oplus, \leq)$ be a local preordered monoid. We can extend this structure to $(S(\mathcal{Y}), \oplus, \preceq)$ by defining for $\mathcal{M}_1, \mathcal{M}_2 \in S(\mathcal{Y})$:

$$\mathcal{M}_1 \oplus \mathcal{M}_2 := \{M_1 \oplus M_2 : M_1 \in \mathcal{M}_1, M_2 \in \mathcal{M}_2\},$$
$$\mathcal{M}_1 \preceq \mathcal{M}_2 \text{ if for every } M_2 \in \mathcal{M}_2 \text{ there is } M_1 \in \mathcal{M}_1 \text{ with } M_1 \leq M_2.$$

Lemma 2.31. *Let $(\mathcal{Y}, \oplus, \leq)$ be a local preordered monoid with neutral element $\theta \in \mathcal{Y}$. Then, $(S(\mathcal{Y}), \oplus, \preccurlyeq)$ is a local preordered monoid as well with neutral element $\{\theta\}$.*

Proof. The first thing to verify is $\oplus\colon S(\mathcal{X}) \times S(\mathcal{X}) \to S(\mathcal{X})$ being a local function. Note to this end that $\mathcal{M}_1 \oplus \mathcal{M}_2 \in S(\mathcal{X})$. Indeed, consider a partition $(A_n)_{n\in\mathbb{N}}$ and a sequence $(M_n^1 \oplus M_n^2)_{n\in\mathbb{N}} \in \mathcal{M}_1 \oplus \mathcal{M}_2$. Since $\oplus$ is local, it follows that $\sum_{n\in\mathbb{N}} 1_{A_n}(M_n^1 \oplus M_n^2) = (\sum_{n\in\mathbb{N}} 1_{A_n} M_n^1) \oplus (\sum_{n\in\mathbb{N}} 1_{A_n} M_n^2)$. The latter is an element in $\mathcal{M}_1 \oplus \mathcal{M}_2$, since $\mathcal{M}_1$ and $\mathcal{M}_2$ are σ-stable. Now consider a partition $(A_n)_{n\in\mathbb{N}}$ and sequences $(\mathcal{M}_n^1)_{n\in\mathbb{N}}$ and $(\mathcal{M}_n^2)_{n\in\mathbb{N}}$. It holds that

$$\begin{aligned}
\sum_{n\in\mathbb{N}} 1_{A_n}(\mathcal{M}_n^1 \oplus \mathcal{M}_n^2) &= \sum_{n\in\mathbb{N}} 1_{A_n}\{M_n^1 \oplus M_n^2 : M_n^1 \in \mathcal{M}_n^1, M_n^2 \in \mathcal{M}_n^2\} \\
&= \left\{ M^1 \oplus M^2 : M^1 \in \sum_{n\in\mathbb{N}} 1_{A_n}\mathcal{M}_n^1, M^2 \in \sum_{n\in\mathbb{N}} 1_{A_n}\mathcal{M}_n^2 \right\} \\
&= \left(\sum_{n\in\mathbb{N}} 1_{A_n}\mathcal{M}_n^1 \right) \oplus \left(\sum_{n\in\mathbb{N}} 1_{A_n}\mathcal{M}_n^2 \right),
\end{aligned}$$

where we used locality of $\oplus$.

Next, we show that $\preceq$ is a local relation. Consider a partition $(A_n)_{n\in\mathbb{N}}$ and sequences $(\mathcal{M}_n^1)_{n\in\mathbb{N}}$ and $(\mathcal{M}_n^2)_{n\in\mathbb{N}}$ such that $\mathcal{M}_n^1 \preceq \mathcal{M}_n^2$ for any $n \in \mathbb{N}$. That is, for any $M_n^2 \in \mathcal{M}_n^2$ there exists $M_n^1 \in \mathcal{M}_n^1$ with $M_n^1 \leq M_n^2$. Consider the sets $\mathcal{M}^1 = \sum_{n\in\mathbb{N}} 1_{A_n}\mathcal{M}_n^1$ and $\mathcal{M}^2 = \sum_{n\in\mathbb{N}} 1_{A_n}\mathcal{M}_n^2$ which are both in $S(\mathcal{X})$. Pick $M^2 \in \mathcal{M}^2$ which has to be of the form $M^2 = \sum_{n\in\mathbb{N}} 1_{A_n} M_n^2$ for some $(M_n^2)_{n\in\mathbb{N}}$, $M_n^2 \in \mathcal{M}_n^2$ for all $n \in \mathbb{N}$. Choosing the corresponding M_n^1 as above and using the locality of $\leq$ yields $M^1 := \sum_{n\in\mathbb{N}} 1_{A_n} M_n^1 \leq \sum_{n\in\mathbb{N}} 1_{A_n} M_n^2 = M^2$, where $M^1 \in \mathcal{M}^1$. Thus, $\mathcal{M}^1 \preceq \mathcal{M}^2$ which shows that $\preceq$ is local.

Clearly $\preceq$ is transitive and reflexive. Moreover, $\{\theta\}$ is the neutral element of $\oplus$. To show the Property (Add), consider $\mathcal{M}_1 \preceq \mathcal{M}_2$, some $\mathcal{M} \in S(\mathcal{X})$ and an arbitrary $P \in \mathcal{M}_2 \oplus \mathcal{M}$. There exists $M_2 \in \mathcal{M}_2$ and $M \in \mathcal{M}$ with $P = M_2 \oplus M$. As $\mathcal{M}_1 \preceq \mathcal{M}_2$, there exists $M_1 \in \mathcal{M}_1$ with $M_1 \leq M_2$. By (Add) of $\oplus$, it follows that $M_1 \oplus M \leq M_2 \oplus M = P$. As $M_1 \oplus M \in \mathcal{M}_1$, we showed $\mathcal{M}_1 \oplus \mathcal{M} \preceq \mathcal{M}_2 \oplus \mathcal{M}$. $\square$

2.3.3 Kirk-Caristi Fixed Point Theorem

The Kirk-Caristi fixed point theorem in a deterministic setting was proven by Caristi and Kirk (compare [60]). It states that on a complete metric space (X, d) a lower semicontinuous function $T : X \to \mathbb{R} \cup \{\infty\}$, with $\inf f \in \mathbb{R}$, and fulfilling $d(x, T(x)) \leq$

$f(x) - f(T(x))$ has a fixed point.

We consider a local set-valued map $T : \mathcal{X} \to S(\mathcal{X})$. Recall the definitions of a fixed point and an invariant point of T given in Definition 2.9.

Corollary 2.32. *Let the assumptions of Theorem 2.25 be in force. If, additionally, the map $T : \mathcal{X} \to S(\mathcal{X})$ satisfies the condition*

$$\forall X \in \mathcal{X},\ \exists X' \in T(X) :\ f(X') \oplus \Phi(X', X) \leq f(X), \tag{WC}$$

then T has a fixed point.

If the map $T : \mathcal{X} \to S(\mathcal{X})$ satisfies

$$\forall X \in \mathcal{X},\ \forall X' \in T(X) :\ f(X') \oplus \Phi(X', X) \leq f(X), \tag{SC}$$

then T has an invariant point.

Proof. By Theorem 2.25, there is $\bar{X} \in \mathcal{X}$ such that

$$f(X) \oplus \Phi(X, \bar{X}) \leq f(\bar{X}) \quad \Longrightarrow \quad X = \bar{X}.$$

Hence, $\bar{X}$ is the only point X' which satisfies (WC) or (SC) and thereby $\bar{X} \in T(\bar{X})$ or $T(\bar{X}) = \{\bar{X}\}$, respectively, which proves the corollary. $\square$

Conversely, Theorem 2.25 may also be proven using the fixed point result above. Indeed, assume that *(b)* of Theorem 2.25 does not hold, that is

$$\forall X \in \mathcal{X},\ \exists X' \neq X :\ f(X') \oplus \Phi(X', X) \leq f(X).$$

The map $T : \mathcal{X} \to S(\mathcal{X})$ satisfies (SC) and has no invariant point, meaning the assertions of Corollary 2.32 can not hold. In this sense, the two corollaries are equivalent.

We again give the particular case of the L^0-version of the Kirk-Caristi fixed point theorem.

Corollary 2.33. *Let $(\mathcal{X}, d)$ a complete L^0-metric module and $f : \mathcal{X} \to L^0$, $\operatorname{ess\,inf} f \in L^0$, a lower semicontinuous function fulfilling*

$$\forall X \in \mathcal{X},\ \exists X' \in \mathcal{X} : f(X') + d(X', X) \leq f(X).$$

Then f has a fixed point.

Proof. The proof is a consequence of plugging $\phi = d$ and $\mathcal{Y} = L^0$ into the previous theorem. $\square$

2.3.4 Takahashi's Minimization Theorem

Takahashi's minimization theorem [68] deals with lower semicontionuous functions $f : X \to \mathbb{R} \cup \{+\infty\}$ on a complete metric space (X, d) for which holds $\inf f \in \mathbb{R}$. If for each $y \in X$ with $\inf f < f(y)$ there exists $x \in X$ with $x \neq y$ and $f(x) + d(x, y) \leq d(y)$, then there exists $x_0 \in X$ with $f(x_0) = \inf f$, that is there is a minimizer of f in X. Its equivalence to Ekeland's variational principle has been observed in [65] and [58]. We will prove a transfer of this minimization theorem to the L^0-setting.

Definition 2.34. The set $\min \mathcal{Y} := \{Y \in \mathcal{Y} : \forall X \in \mathcal{Y} \text{ with } X \leq Y \text{ it follows } Y \leq X\}$ denotes the set of minimal elements.

The set $\min \mathcal{Y}$ is in $S(\mathcal{Y})$. Indeed, consider a partition $(A_n)_{n\in\mathbb{N}}$ and $(Y_n)_{n\in\mathbb{N}} \subseteq \min \mathcal{Y}$. Then, for $X \leq \sum_{n\in\mathbb{N}} 1_{A_n} Y_n$ it holds that $1_{A_n} X + 1_{A_n^c} Y_n \leq Y_n$ for all $n \in \mathbb{N}$. This implies $Y_n \leq 1_{A_n} X + 1_{A_n^c} Y$ for all $n \in \mathbb{N}$ and thereby $\sum_{n\in\mathbb{N}} 1_{A_n} Y_n \leq X$. This shows that $\sum_{n\in\mathbb{N}} 1_{A_n} Y_n \in \min \mathcal{Y}$.

Corollary 2.35. *Let the assumptions of Theorem 2.25 be fulfilled. Assume in addition that*

$$X_1, X_2 \in \mathcal{X},\ f(X_1) \leq f(X_2),\ f(X_2) \not\leq f(X_1)$$

implies

$$\exists X_3 \in \mathcal{X} :\ X_3 \neq X_2,\ f(X_3) \oplus \Phi(X_3, X_2) \leq f(X_2).$$

Then, there exists $\bar{X} \in \mathcal{X}$ such that $f(\bar{X}) \in \min f(\mathcal{X})$.

Proof. By Theorem 2.25, there is $\bar{X} \in \mathcal{X}$ such that

$$f(X) \oplus \Phi(X, \bar{X}) \leq f(\bar{X}) \quad \Longrightarrow \quad X = \bar{X}. \tag{2.5}$$

Suppose that $f(\bar{X}) \notin \min f(\mathcal{X})$. Then, there exists $W \in \mathcal{X}$ such that $f(W) \leq f(\bar{X})$ but $f(\bar{X}) \not\leq f(W)$. By the additional assumption, there is some $X_3 \neq \bar{X}$ with $f(X_3) \oplus \Phi(X_3, \bar{X}) \leq f(\bar{X})$. This contradicts (2.5). □

Corollary 2.36. *Let $(\mathcal{X}, d)$ be a complete L^0-metric module and $f : \mathcal{X} \to \overline{L^0}$ a local, lower semicontinuous function, with $\operatorname{ess\,inf} f \in L^0$. If for each $Y \in \mathcal{X}$ with $\operatorname{ess\,inf} f < f(Y)$ there exists $X \in \mathcal{X}$ with $X \neq Y$ and $f(X) + d(X, Y) \leq d(Y)$, then there exists $X_0 \in \mathcal{X}$ with $f(X_0) = \operatorname{ess\,inf} f$.*

Proof. Again, setting $\phi = d$ and $\mathcal{Y} = L^0$ within the previous theorem yields the claim. □

Suppose now that Φ is symmetric meaning $\Phi(X, Y) = \Phi(Y, X)$ for all $X, Y \in \mathcal{X}$. In this case, Theorem 2.25 can be proven using Corollary 2.35. Indeed, assume that *(b)* of Theorem 2.25 does not hold, that is for every $X \in \mathcal{X}$ there exists some $X' \neq X$ such that $f(X') \oplus \Phi(X', X) \leq f(X)$. Let $\bar{X} \in \mathcal{X}$ such that $f(\bar{X}) \in \min f(\mathcal{X})$, that is $X \in \mathcal{X}$, $f(X) \leq f(\bar{X})$ implies that $f(\bar{X}) \leq f(X)$. An $\bar{X}$ with these properties does exist by Corollary 2.35. By assumption, there is also $\bar{X}' \in \mathcal{X}$ such that $f(\bar{X}') \oplus \Phi(\bar{X}', \bar{X}) \leq f(\bar{X})$. From $\theta \leq \Phi(\bar{X}', \bar{X})$, we obtain $f(\bar{X}') \leq f(\bar{X}') \oplus \Phi(\bar{X}', \bar{X})$. The transitivity of $\leq$ implies $f(\bar{X}') \leq f(\bar{X})$ and therefore the minimality of $f(\bar{X})$ gives $f(\bar{X}) \leq f(\bar{X}')$. Using this and symmetry of Φ, we conclude

$$f(\bar{X}) \oplus \Phi(\bar{X}, \bar{X}') = f(\bar{X}) \oplus \Phi(\bar{X}', \bar{X}) \leq f(\bar{X}') \oplus \Phi(\bar{X}', \bar{X}) \leq f(\bar{X}) \leq f(\bar{X}').$$

Since the corresponding $\preceq$ is antisymmetric (this is due to the regularity of Φ, compare Lemma 2.8), it holds that $\bar{X}' = \bar{X}$, a contradiction.

3 Brouwer Fixed Point Theorem in $(L^0)^d$

In this part of the thesis we examine a further application of L^0-theory. It corresponds to the paper "Brouwer fixed point theorem in $(L^0)^d$" by Drapeau, Karliczek, Kupper, and Streckfuß [32]. We establish a translation of the Brouwer fixed point theorem to functions in $(L^0)^d$. We define a simplex in that context and prove that every local, sequentially continuous function has a fixed point. To do so, we first prove a result similar to Sperner's lemma. From the measurable structure of the problem, it turns out that we have to work with local, measurable labeling functions. To cope with this difficulty and to maintain certain uniform properties, we subdivide the conditional simplex barycentrically. We then prove the existence of a measurable completely labeled conditional simplex, contained in the original one, which turns out to be a suitable σ-combination of elements of the barycentric subdivision along a partition of Ω. Thus, we can construct a sequence of conditional simplexes converging to an element. We then show that this element has to be a fixed point which is measurable by construction. With the fixed point result for conditional simplexes at hand, we can prove the theorem also for arbitrary closed L^0-convex sets.

This chapter is organized as follows. In the first section, we present the basic concepts concerning $(L^0)^d$ as an L^0-module. We define conditional simplexes and examine their basic properties. In the second section, we define measurable labeling functions and show the Brouwer fixed point theorem for conditional simplexes via a construction in the spirit of Sperner's lemma. In the third section, we show a fixed point result for L^0-convex, bounded and sequentially closed sets in $(L^0)^d$. With this result at hand, we present the topological implications known from the real-valued case. On the one hand, we show the impossibility of contracting a ball to a sphere in $(L^0)^d$ and on the other hand, prove an intermediate value theorem in L^0.

3.1 Conditional Simplex

For a probability space $(\Omega, \mathcal{A}, P)$, let $L^0 = L^0(\Omega, \mathcal{A}, P)$. Recall, that for $X, Y \in L^0$, the relations $X \geq Y$ and $X > Y$ have to be understood P-almost surely. The set L^0 with the P-almost everywhere order is a lattice ordered ring and for a nonempty subset

$\mathcal{C} \subseteq L^0$ we denote the least upper bound by $\operatorname{ess\,sup} \mathcal{C}$ and the greatest lower bound by $\operatorname{ess\,inf} \mathcal{C}$, respectively (compare [24]). For $m \in \mathbb{R}$, we denote the constant random variable $m 1_\Omega$ by m. The set of random variables with values in a set $M \subseteq \mathbb{R}$ is denoted by $M(\mathcal{A})$. For example, $\{1, \ldots, r\}(\mathcal{A})$ is the set of $\mathcal{A}$-measurable functions with values in $\{1, \ldots, r\} \subseteq \mathbb{N}$, $[0,1](\mathcal{A}) = \{Z \in L^0 \colon 0 \leq Z \leq 1\}$ and $(0,1)(\mathcal{A}) = \{Z \in L^0 \colon 0 < Z < 1\}$.

The convex hull of $X_1, \ldots, X_N \in (L^0)^d$, $N \in \mathbb{N}$, is defined as

$$\operatorname{conv}(X_1, \ldots, X_N) = \left\{ \sum_{i=1}^N \lambda_i X_i \colon \lambda_i \in L^0_+, \sum_{i=1}^N \lambda_i = 1 \right\}.$$

An element $Y = \sum_{i=1}^N \lambda_i X_i$ such that $\lambda_i > 0$ for all $i \in I \subseteq \{1, \ldots, N\}$ is called a strict convex combination of $\{X_i \colon i \in I\}$. Moreover, a set $\mathcal{C} \subseteq (L^0)^d$ is said to be L^0-convex if for any $X, Y \in \mathcal{C}$ and $\lambda \in [0,1](\mathcal{A})$ it holds that $\lambda X + (1-\lambda) Y \in \mathcal{C}$.

For $\mathcal{X}, \mathcal{Y} \subseteq (L^0)^d$, we call a function $f \colon \mathcal{X} \to \mathcal{Y}$ sequentially continuous if for every sequence $(X_n)_{n \in \mathbb{N}}$ in $\mathcal{X}$ converging to $X \in \mathcal{X}$ P-almost-surely it holds that $f(X_n)$ converges to $f(X)$ P-almost surely. Further, the L^0-scalar product and L^0-norm on $(L^0)^d$ are defined as

$$\langle X, Y \rangle = \sum_{i=1}^d X_i Y_i \quad \text{and} \quad \|X\| = \langle X, X \rangle^{\frac{1}{2}}.$$

We call $\mathcal{C} \subseteq (L^0)^d$ bounded if $\operatorname{ess\,sup}_{X \in \mathcal{C}} \|X\| \in L^0$ and sequentially closed if it contains all P-almost sure limits of sequences in $\mathcal{C}$. Further, the diameter of $\mathcal{C} \subseteq (L^0)^d$ is defined as $\operatorname{diam}(\mathcal{C}) = \operatorname{ess\,sup}_{X, Y \in \mathcal{C}} \|X - Y\|$.

Definition 3.1. Elements $X_1, \ldots, X_N$ of $(L^0)^d$, $N \in \mathbb{N}$, are said to be affinely independent if either $N = 1$ or $N > 1$ and $\{X_i - X_N\}_{i=1}^{N-1}$ are linearly independent, that is

$$\sum_{i=1}^{N-1} \lambda_i (X_i - X_N) = 0 \quad \text{implies} \quad \lambda_1 = \cdots = \lambda_{N-1} = 0, \tag{3.1}$$

where $\lambda_1, \ldots, \lambda_{N-1} \in L^0$.

The definition of affine independence is equivalent to

$$\sum_{i=1}^N \lambda_i X_i = 0 \;\text{ and }\; \sum_{i=1}^N \lambda_i = 0 \quad \text{implies} \quad \lambda_1 = \cdots = \lambda_N = 0. \tag{3.2}$$

Indeed, first we show that (3.1) implies (3.2). Let $\sum_{i=1}^N \lambda_i X_i = 0$ and $\sum_{i=1}^N \lambda_i = 0$. Then $\sum_{i=1}^{N-1} \lambda_i (X_i - X_N) = \lambda_N X_N + \sum_{i=1}^{N-1} \lambda_i X_i = 0$. By assumption (3.1), $\lambda_1 = \cdots = \lambda_{N-1} = 0$, thus also $\lambda_N = 0$. To see that (3.2) implies (3.1), let $\sum_{i=1}^{N-1} \lambda_i (X_i - X_N) = 0$. With $\lambda_N = -\sum_{i=1}^{N-1} \lambda_i$, it holds $\sum_{i=1}^N \lambda_i X_i = \lambda_N X_N + \sum_{i=1}^{N-1} \lambda_i X_i = \sum_{i=1}^{N-1} \lambda_i (X_i -$

$X_N) = 0$. By assumption (3.2), $\lambda_1 = \dots = \lambda_N = 0$.

Remark 3.2. We observe that if $(X_i)_{i=1}^N \subseteq (L^0)^d$ are affinely independent, then $(\lambda X_i)_{i=1}^N$ for $\lambda \in L^0_{++}$ and $(X_i + Y)_{i=1}^N$ for $Y \in (L^0)^d$ are affinely independent. Moreover, if a family $X_1, \dots, X_N$ is affinely independent, then also $1_B X_1, \dots, 1_B X_N$ are affinely independent on $B \in \mathcal{A}_+$, which means from $\sum_{i=1}^N 1_B \lambda_i X_i = 0$ and $\sum_{i=1}^N 1_B \lambda_i = 0$ it always follows that $1_B \lambda_i = 0$ for all $i = 1, \dots, N$.

Definition 3.3. A conditional simplex in $(L^0)^d$ is a set of the form

$$\mathcal{S} = \operatorname{conv}(X_1, \dots, X_N)$$

such that $X_1, \dots, X_N \in (L^0)^d$ are affinely independent. We call $N \in \mathbb{N}$ the dimension of $\mathcal{S}$.

Remark 3.4. In a conditional simplex $\mathcal{S} = \operatorname{conv}(X_1, \dots, X_N)$, the coefficients of convex combinations are unique in the sense that

$$\sum_{i=1}^N \lambda_i X_i = \sum_{i=1}^N \mu_i X_i \text{ and } \sum_{i=1}^N \lambda_i = \sum_{i=1}^N \mu_i = 1 \quad \text{implies} \quad \lambda_i = \mu_i \text{ for all } i = 1, \dots, N. \tag{3.3}$$

Indeed, since $\sum_{i=1}^N (\lambda_i - \mu_i) X_i = 0$ and $\sum_{i=1}^N (\lambda_i - \mu_i) = 0$, it follows from (3.2) that $\lambda_i - \mu_i = 0$ for all $i = 1, \dots, N$.

Remark 3.5. Note that the present setting - L^0-modules and the sequential P-almost sure convergence - is of local nature. This is, for instance, not the case for subsets of L^p or the convergence in the L^p-norm for $1 \leq p < \infty$. First, L^p is not closed under multiplication and hence neither a ring nor a module over itself, so that we cannot even speak about affine independence. Second, it is in general not a σ-stable subspace of L^0. However, for a conditional simplex $\mathcal{S} = \operatorname{conv}(X_1, \dots, X_N)$ in $(L^0)^d$ such that any X_k is in $(L^p)^d$, it holds that $\mathcal{S}$ is uniformly bounded by $N \sup_{k=1,\dots,N} \|X_k\| \in L^p$. This uniform boundedness yields that any P-almost sure converging sequence in $\mathcal{S}$ is also converging in the L^p-norm for $1 \leq p < \infty$ due to the dominated convergence theorem. This shows how one can translate results from L^0 to L^p.

Since a conditional simplex is a convex hull of finitely many element, it is in particular σ-stable. In contrast to a simplex in $\mathbb{R}^d$, the representation of $\mathcal{S}$ as a convex hull of affinely independent elements is unique but up to σ-stability.

Proposition 3.6. *Let $(X_i)_{i=1}^N$ and $(Y_i)_{i=1}^N$ be families in $(L^0)^d$ with $\sigma(X_1, \dots, X_N) = \sigma(Y_1, \dots, Y_N)$. Then $\operatorname{conv}(X_1, \dots, X_N) = \operatorname{conv}(Y_1, \dots, Y_N)$. Moreover, $(X_i)_{i=1}^N$ are affinely independent if and only if $(Y_i)_{i=1}^N$ are affinely independent.*

If $\mathcal{S}$ is a conditional simplex such that $\mathcal{S} = \operatorname{conv}(X_1, \dots, X_N) = \operatorname{conv}(Y_1, \dots, Y_N)$, then it holds that $\sigma(X_1, \dots, X_N) = \sigma(Y_1, \dots, Y_N)$.

Proof. Suppose $\sigma(X_1, \dots, X_N) = \sigma(Y_1, \dots, Y_N)$. For $i = 1, \dots, N$, it holds that

$$X_i \in \sigma(X_1, \dots, X_N) = \sigma(Y_1, \dots, Y_N) \subseteq \operatorname{conv}(Y_1, \dots, Y_N).$$

Therefore, $\operatorname{conv}(X_1, \dots, X_N) \subseteq \operatorname{conv}(Y_1, \dots, Y_N)$ and the reverse inclusion holds analogously.

Now, let $(X_i)_{i=1}^N$ be affinely independent and $\sigma(X_1, \dots, X_N) = \sigma(Y_1, \dots, Y_N)$. We want to show that $(Y_i)_{i=1}^N$ are affinely independent. To that end, we define the affine hull

$$\operatorname{aff}(X_1, \dots, X_N) = \left\{ \sum_{i=1}^N \lambda_i X_i \colon \lambda_i \in L^0, \sum_{i=1}^N \lambda_i = 1 \right\}.$$

First, let $Z_1, \dots, Z_M \in (L^0)^d$, $M \in \mathbb{N}$, such that $\sigma(X_1, \dots, X_N) = \sigma(Z_1, \dots, Z_M)$. We show that if $1_A \operatorname{aff}(X_1, \dots, X_N) \subseteq 1_A \operatorname{aff}(Z_1, \dots, Z_M)$ for $A \in \mathcal{A}_+$ and $X_1, \dots, X_N$ are affinely independent then $M \geq N$. Since $X_i \in \sigma(X_1, \dots, X_N) = \sigma(Z_1, \dots, Z_M) \subseteq \operatorname{aff}(Z_1, \dots, Z_M)$, we have $\operatorname{aff}(X_1, \dots, X_N) \subseteq \operatorname{aff}(Z_1, \dots, Z_M)$. Further, it holds that $X_1 = \sum_{i=1}^M 1_{B_i^1} Z_i$ for a partition $(B_i^1)_{i=1}^M$ and hence there exists at least one $B_{k_1}^1$ such that $A_{k_1}^1 := B_{k_1}^1 \cap A \in \mathcal{A}_+$, and $1_{A_{k_1}^1} X_1 = 1_{A_{k_1}^1} Z_{k_1}$. Therefore,

$$1_{A_{k_1}^1} \operatorname{aff}(X_1, \dots, X_N) \subseteq 1_{A_{k_1}^1} \operatorname{aff}(Z_1, \dots, Z_M) = 1_{A_{k_1}^1} \operatorname{aff}(\{X_1, Z_1, \dots, Z_M\} \setminus \{Z_{k_1}\}).$$

For $X_2 = \sum_{i=1}^M 1_{A_i^2} Z_i$, we find a set A_k^2 such that $A_{k_2}^2 = A_k^2 \cap A_{k_1}^1 \in \mathcal{A}_+$, $1_{A_{k_2}^2} X_2 = 1_{A_{k_2}^2} Z_{k_2}$ and $k_1 \neq k_2$. Assume to the contrary $k_2 = k_1$, then there exists a set $B \in \mathcal{A}_+$ such that $1_B X_1 = 1_B X_2$, which is a contradiction to the affine independence of $(X_i)_{i=1}^N$. Hence, we can again substitute Z_{k_2} by X_2 on $A_{k_2}^2$. Inductively, we find $k_1, \dots, k_N$ such that

$$1_{A_{k_N}} \operatorname{aff}(X_1, \dots, X_N) \subseteq 1_{A_{k_N}} \operatorname{aff}(\{X_1, \dots, X_N, Z_1, \dots, Z_M\} \setminus \{Z_{k_1}, \dots Z_{k_N}\})$$

which shows $M \geq N$. Now suppose that $Y_1, \dots, Y_N$ are not affinely independent. This means that there exist $(\lambda_i)_{i=1}^N$ such that $\sum_{i=1}^N \lambda_i Y_i = \sum_{i=1}^N \lambda_i = 0$ but not all coefficients λ_i are zero, without loss of generality, $\lambda_1 > 0$ on $A \in \mathcal{A}_+$. Thus, $1_A Y_1 = -1_A \sum_{i=2}^N \frac{\lambda_i}{\lambda_1} Y_i$ and it holds that $1_A \operatorname{aff}(Y_1, \dots, Y_N) = 1_A \operatorname{aff}(Y_2, \dots, Y_N)$. To see this, consider $1_A Z = 1_A \sum_{i=1}^N \mu_i Y_i \in 1_A \operatorname{aff}(Y_1, \dots, Y_N)$, which means $1_A \sum_{i=1}^N \mu_i = 1_A$. Thus, inserting for Y_1,

$$1_A Z = 1_A \left[\sum_{i=2}^N \mu_i Y_i - \mu_1 \sum_{i=2}^N \frac{\lambda_i}{\lambda_1} Y_i \right] = 1_A \left[\sum_{i=2}^N \left(\mu_i - \mu_1 \frac{\lambda_i}{\lambda_1} \right) Y_i \right].$$

Moreover,

$$1_A \left[\sum_{i=2}^N \left(\mu_i - \mu_1 \frac{\lambda_i}{\lambda_1} \right) \right] = 1_A \left[\sum_{i=2}^N \mu_i \right] + 1_A \left[-\frac{\mu_1}{\lambda_1} \sum_{i=2}^N \lambda_i \right] = 1_A(1 - \mu_1) + 1_A \frac{\mu_1}{\lambda_1} \lambda_1 = 1_A.$$

Hence, $1_A Z \in 1_A \operatorname{aff}(Y_2, \dots, Y_N)$. It follows that

$$1_A \operatorname{aff}(X_1, \dots, X_N) = 1_A \operatorname{aff}(Y_1, \dots, Y_N) = 1_A \operatorname{aff}(Y_2, \dots, Y_N).$$

This is a contradiction to the former part of the proof (because $N - 1 \ngeq N$).

Next, we show that in a conditional simplex $\mathcal{S} = \operatorname{conv}(X_1, \dots, X_N)$ it holds that $X \in \sigma(X_1, \dots, X_N)$ if and only if there do not exist Y and Z in $\mathcal{S} \setminus \{X\}$ and $\lambda \in (0,1)(\mathcal{A})$ such that $\lambda Y + (1-\lambda)Z = X$. Consider $X \in \sigma(X_1, \dots, X_N)$ which is $X = \sum_{k=1}^N 1_{A_k} X_k$ for a partition $(A_k)_{k=1,\dots,N}$. Now assume to the contrary that we find $Y = \sum_{k=1}^N \lambda_k X_k$ and $Z = \sum_{k=1}^N \mu_k X_k$ in $\mathcal{S} \setminus \{X\}$ such that $X = \lambda Y + (1-\lambda)Z$. This means that $X = \sum_{k=1}^N (\lambda \lambda_k + (1-\lambda)\mu_k) X_k$. Due to uniqueness of the coefficients (compare (3.3)) in a conditional simplex, we have $\lambda\lambda_k + (1-\lambda)\mu_k = 1_{A_k}$ for all $k = 1 \dots, N$. By means of $0 < \lambda < 1$, it holds that $\lambda\lambda_k + (1-\lambda)\mu_k = 1_{A_k}$ if and only if $\lambda_k = \mu_k = 1_{A_k}$. Since the last equality holds for all k, it follows that $Y = Z = X$. Therefore, we cannot find Y and Z in $\mathcal{S} \setminus \{X\}$ such that X is a strict convex combination of them. Reversely, consider $X \in \mathcal{S}$ such that $X \notin \sigma(X_1, \dots, X_N)$. This means $X = \sum_{k=1}^N \nu_k X_k$ such that there exist ν_{k_1} and ν_{k_2} and $B \in \mathcal{A}_+$ with $0 < \nu_{k_1} < 1$ on B and $0 < \nu_{k_2} < 1$ on B. Define $\varepsilon := \operatorname{ess\,inf}\{\nu_{k_1}, \nu_{k_2}, 1 - \nu_{k_1}, 1 - \nu_{k_2}\}$. Then define $\mu_k = \lambda_k = \nu_k$ if $k_1 \neq k \neq k_2$ and $\lambda_{k_1} = \nu_{k_1} - \varepsilon$, $\lambda_{k_2} = \nu_{k_2} + \varepsilon$, $\mu_{k_1} = \nu_{k_1} + \varepsilon$ and $\mu_{k_2} = \nu_{k_2} - \varepsilon$. Thus, $Y = \sum_{k=1}^N \lambda_k X_k$ and $Z = \sum_{k=1}^N \mu_k X_k$ fulfill $0.5Y + 0.5Z = X$ but both are not equal to X by construction. Hence, X can be written as a strict convex combination of elements in $\mathcal{S} \setminus \{X\}$. To conclude, consider $X \in \sigma(X_1, \dots, X_N) \subseteq \mathcal{S} = \operatorname{conv}(X_1, \dots, X_N) = \operatorname{conv}(Y_1, \dots, Y_N)$. Since $X \in \sigma(X_1, \dots, X_N)$, it is not a strict convex combination of elements in $\mathcal{S} \setminus \{X\}$, in particular, of elements in $\operatorname{conv}(Y_1, \dots, Y_N) \setminus \{X\}$. Therefore, X is also in $\sigma(Y_1, \dots, Y_N)$. Hence, $\sigma(X_1, \dots, X_N) \subseteq \sigma(Y_1, \dots, Y_N)$. With the same argumentation, the other inclusion follows. □

As an example, let us consider $[0,1](\mathcal{A})$. For an arbitrary $A \in \mathcal{A}$, it holds that 1_A and 1_{A^c} are affinely independent and $\operatorname{conv}(1_A, 1_{A^c}) = \{\lambda 1_A + (1-\lambda) 1_{A^c} \colon 0 \leq \lambda \leq 1\} = [0,1](\mathcal{A})$. Thus, the conditional simplex $[0,1](\mathcal{A})$ can be written as a convex combination of different affinely independent elements of L^0. This is due to the fact that $\sigma(0,1) = \{1_B : B \in \mathcal{A}\} = \sigma(1_A, 1_{A^c})$ for all $A \in \mathcal{A}$.

Remark 3.7. In $(L^0)^d$, let e_i be the random variable which is 1 in the ith component and 0 in any other. Then the family $0, e_1, \dots, e_d$ is affinely independent and $(L^0)^d = \operatorname{aff}(0, e_1, \dots, e_d)$. Hence, the maximal number of affinely independent elements in $(L^0)^d$ is $d+1$.

The characterization of $X \in \sigma(X_1, \dots, X_N)$ leads to the following definition.

Definition 3.8. Let $\mathcal{S} = \operatorname{conv}(X_1, \dots, X_N)$ be a conditional simplex. We define the set of extremal points $\operatorname{ext}(\mathcal{S}) = \sigma(X_1, \dots, X_N)$. For an index set I and a collection $\mathscr{S} = (\mathcal{S}_i)_{i \in I}$ of conditional simplexes, we denote $\operatorname{ext}(\mathscr{S}) = \sigma(\cup_{i \in I} \operatorname{ext}(\mathcal{S}_i))$.

Remark 3.9. Let $\mathcal{S}^j = \operatorname{conv}(X_1^j, \dots, X_N^j)$, $j \in \mathbb{N}$, be conditional simplexes of the same dimension N and $(A_j)_{j\in\mathbb{N}}$ a partition. Then $\sum_{j\in\mathbb{N}} 1_{A_j}\mathcal{S}^j$ is again a conditional simplex. To that end, we define $Y_k = \sum_{j\in\mathbb{N}} 1_{A_j} X_k^j$ and recognize $\sum_{j\in\mathbb{N}} 1_{A_j}\mathcal{S}^j = \operatorname{conv}(Y_1, \dots, Y_N)$. Indeed,

$$\sum_{k=1}^N \lambda_k Y_k = \sum_{k=1}^N \lambda_k \sum_{j\in\mathbb{N}} 1_{A_j} X_k^j = \sum_{j\in\mathbb{N}} 1_{A_j} \sum_{k=1}^N \lambda_k X_k^j \in \sum_{j\in\mathbb{N}} 1_{A_j}\mathcal{S}^j, \tag{3.4}$$

shows $\operatorname{conv}(Y_1, \dots, Y_N) \subseteq \sum_{j\in\mathbb{N}} 1_{A_j}\mathcal{S}^j$. The other inclusion follows by considering $\sum_{k=1}^N \lambda_k^j X_k^j \in \mathcal{S}^j$ and defining $\lambda_k = \sum_{j\in\mathbb{N}} 1_{A_j}\lambda_k^j$. To show that $Y_1, \dots, Y_N$ are affinely independent, we consider $\sum_{k=1}^N \lambda_k Y_k = 0 = \sum_{k=1}^N \lambda_k$. Then by (3.4) it holds that $1_{A_j}\sum_{k=1}^N \lambda_k X_k^j = 0$ and since $\mathcal{S}^j$ is a conditional simplex, $1_{A_j}\lambda_k = 0$ for all $j \in \mathbb{N}$ and $k = 1, \dots, N$. From the fact that $(A_j)_{j\in\mathbb{N}}$ is a partition, it follows that $\lambda_k = 0$ for all $k = 1, \dots, N$.

We will prove the Brouwer fixed point theorem in the present setting using an L^0-module version of Sperner's lemma. As in the unconditional case, we have to subdivide a conditional simplex into smaller ones. For our argumentation, we cannot use arbitrary subdivisions and need very special properties of the conditional simplexes in which we subdivide. This leads to the following definition.

Definition 3.10. Let $\mathcal{S} = \operatorname{conv}(X_1, \dots, X_N)$ be a conditional simplex and $\mathrm{S_N}$ be the group of permutations of $\{1, \dots, N\}$. For $\pi \in \mathrm{S_N}$, we define

$$\begin{aligned} Y_k^\pi &= \frac{1}{k}\sum_{i=1}^k X_{\pi(i)}, \quad k = 1, \dots, N, \\ \mathcal{C}_\pi &= \operatorname{conv}\left(Y_1^\pi, \dots, Y_N^\pi\right). \end{aligned}$$

We call $(\mathcal{C}_\pi)_{\pi\in\mathrm{S_N}}$ the barycentric subdivision of $\mathcal{S}$.

Lemma 3.11. *Let the elements $X_1, \dots, X_N \in (L^0)^d$ be affinely independent and $\mathcal{S} = \operatorname{conv}(X_1, \dots, X_N)$. The barycentric subdivision of $\mathcal{S}$ is a collection of finitely many conditional simplexes satisfying the following properties*

(i) $\sigma(\bigcup_{\pi\in\mathrm{S_N}} \mathcal{C}_\pi) = \mathcal{S}$.

(ii) $\mathcal{C}_\pi$ *has dimension* N, $\pi \in \mathrm{S_N}$.

(iii) $\mathcal{C}_\pi \cap \mathcal{C}_{\overline{\pi}}$ *is a conditional simplex of dimension* $r \in \mathbb{N}$ *and* $r < N$ *for* $\pi, \overline{\pi} \in \mathrm{S_N}$, $\pi \neq \overline{\pi}$.

(iv) *For* $s = 1, \dots, N-1$, *let* $\mathcal{B}_s := \operatorname{conv}(X_1, \dots, X_s)$. *All conditional simplexes* $\mathcal{C}_\pi \cap \mathcal{B}_s$, $\pi \in \mathrm{S_N}$, *of dimension* s *subdivide* $\mathcal{B}_s$ *barycentrically.*

Proof. We show the affine independence of $Y_1^\pi, \dots, Y_N^\pi$ in $\mathcal{C}_\pi$. It holds that

$$\lambda_{\pi(1)} X_{\pi(1)} + \lambda_{\pi(2)} \frac{X_{\pi(1)} + X_{\pi(2)}}{2} + \dots + \lambda_{\pi(N)} \frac{\sum_{k=1}^N X_{\pi(k)}}{N} = \sum_{i=1}^N \mu_i X_i,$$

with $\mu_i = \sum_{k=\pi^{-1}(i)}^N \frac{\lambda_{\pi(k)}}{k}$. Since $\sum_{i=1}^N \mu_i = \sum_{i=1}^N \lambda_i$, the affine independence of $Y_1^\pi, \dots, Y_N^\pi$ is obtained by the affine independence of $X_1, \dots, X_N$. Therefore all $\mathcal{C}_\pi$ are conditional simplexes.

As for Condition (i), it clearly holds that $\sigma(\cup_{\pi \in \mathrm{S_N}} \mathcal{C}_\pi) \subseteq \mathcal{S}$. Reversely, let $X = \sum_{i=1}^N \lambda_i X_i \in \mathcal{S}$. Then we find a partition $(A_n)_{n=1,\dots,M}$, for some $M \in \mathbb{N}$, such that on every A_n the indexes are completely ordered, which is $\lambda_{i_1^n} \geq \lambda_{i_2^n} \geq \dots \geq \lambda_{i_N^n}$ on A_n. [1] This means that $X \in 1_{A_n} \mathcal{C}_{\pi^n}$ with $\pi^n(j) = i_j^n$. Indeed, we can rewrite X on A_n as

$$X = (\lambda_{i_1^n} - \lambda_{i_2^n}) X_{i_1^n} + \dots + (N-1)(\lambda_{i_{N-1}^n} - \lambda_{i_N^n}) \frac{\sum_{k=1}^{N-1} X_{i_k^n}}{N-1} + N \lambda_{i_N^n} \frac{\sum_{k=1}^N X_{i_k^n}}{N},$$

which shows that $X \in \mathcal{C}_{\pi^n}$ on A_n. Condition (ii) is fulfilled by construction.

The intersection of two conditional simplexes $\mathcal{C}_\pi$ and $\mathcal{C}_{\overline{\pi}}$ can be expressed in the following manner. Let $J = \{j \colon \{\pi(1), \dots, \pi(j)\} = \{\overline{\pi}(1), \dots, \overline{\pi}(j)\}\}$ be the set of indexes up to which both π and $\overline{\pi}$ have the same set of images. Then

$$\mathcal{C}_\pi \cap \mathcal{C}_{\overline{\pi}} = \operatorname{conv}\left(Y_j^\pi \colon j \in J\right). \tag{3.5}$$

To show $\supseteq$, let $j \in J$. It holds that Y_j^π is in both $\mathcal{C}_\pi$ and $\mathcal{C}_{\overline{\pi}}$ since $\{\pi(1), \dots, \pi(j)\} = \{\overline{\pi}(1), \dots, \overline{\pi}(j)\}$. Since the intersection of L^0-convex sets is L^0-convex, we get this inclusion. As for the reverse inclusion, consider $X \in \mathcal{C}_\pi \cap \mathcal{C}_{\overline{\pi}}$. From $X \in \mathcal{C}_\pi \cap C_{\overline{\pi}}$, it follows that $X = \sum_{i=1}^N \lambda_i (\sum_{k=1}^i \frac{X_{\pi(k)}}{i}) = \sum_{i=1}^N \mu_i (\sum_{k=1}^i \frac{X_{\overline{\pi}(k)}}{i})$. Consider $j \notin J$. By definition of J, there exist $p, q \leq j$ with $\overline{\pi}^{-1}(\pi(p)), \pi^{-1}(\overline{\pi}(q)) \notin \{1, \dots, j\}$. By (3.3), the coefficients of $X_{\pi(p)}$ are equal: $\sum_{i=p}^N \frac{\lambda_i}{i} = \sum_{i=\overline{\pi}^{-1}(\pi(p))}^N \frac{\mu_i}{i}$. The same holds for $X_{\pi(q)}$: $\sum_{i=q}^N \frac{\mu_i}{i} = \sum_{i=\pi^{-1}(\overline{\pi}(q))}^N \frac{\lambda_i}{i}$. Put together

$$\sum_{i=j+1}^N \frac{\mu_i}{i} \leq \sum_{i=q}^N \frac{\mu_i}{i} = \sum_{i=\pi^{-1}(\overline{\pi}(q))}^N \frac{\lambda_i}{i} \leq \sum_{i=j+1}^N \frac{\lambda_i}{i} \leq \sum_{i=p}^N \frac{\lambda_i}{i} = \sum_{i=\overline{\pi}^{-1}(\pi(p))}^N \frac{\mu_i}{i} \leq \sum_{i=j+1}^N \frac{\mu_i}{i}$$

which is only possible if $\mu_j = \lambda_j = 0$ since $p, q \leq j$. Furthermore, if $\mathcal{C}_\pi \cap \mathcal{C}_{\overline{\pi}}$ is of dimension N, by (3.5) it follows that $\pi = \overline{\pi}$. This shows (iii).

Further, for $\mathcal{B}_s = \operatorname{conv}(X_1, \dots, X_s)$, the elements $\mathcal{C}_{\pi'} \cap \mathcal{B}_s$ of dimension s are exactly the ones with $\{\pi'(i) \colon i = 1, \dots, s\} = \{1, \dots, s\}$. To this end, let $C_{\pi'} \cap \mathcal{B}_s$ be of dimension

[1] Let $B_\pi := \{\omega \colon \lambda_{\pi(1)}(\omega) \geq \lambda_{\pi(2)}(\omega) \geq \dots \geq \lambda_{\pi(N)}(\omega)\}$, $\pi \in S_N$. This finite collection of measurable sets fulfills $P(\cup_{\pi \in S_N} B_\pi) = 1$. We can construct a partition $(A_n)_{n=1,\dots,M}$ such that $A_n \subseteq B_{\pi_n}$ for some $\pi_n \in S_N$ and for all $n = 1, \dots, M$. Such a partition fulfills the required property.

s. This means there exists an element Y in this intersection such that $Y = \sum_{i=1}^N \lambda_i X_i$ with $\lambda_i > 0$ for all $i = 1, \dots, s$ and $\lambda_i = 0$ for $i > s$. As an element of $C_{\pi'}$, this Y has a representation of the form $Y = \sum_{j=1}^N (\sum_{k=j}^N \frac{\mu_k}{k}) X_{\pi'(j)}$ for $\sum_{k=1}^N \mu_k = 1$ and $\mu_k \in L^0_+$ for every $k = 1, \dots, N$. Suppose now that there exists some $j_0 \leq s$ with $\pi'(j_0) > s$. Then due to $\lambda_{\pi'(j_0)} = 0$ and the uniqueness of the coefficients (compare (3.3)) in a conditional simplex, it holds that $\sum_{k=j_0}^N \frac{\mu_k}{k} = 0$ and thereby $\sum_{k=j}^N \frac{\mu_k}{k} = 0$ for all $j \geq j_0$. This means $Y = \sum_{j=1}^{j_0-1} (\sum_{k=j}^N \frac{\mu_k}{k}) X_{\pi'(j)}$ and hence Y is the convex combination of $j_0 - 1$ elements with $j_0 - 1 < s$. This contradicts the property that $\lambda_i > 0$ for s elements. Therefore, $(\mathcal{C}_{\pi'} \cap \mathcal{B}_s)_{\pi'}$ is exactly the barycentric subdivision of $\mathcal{B}_s$, which has been shown to fulfill the properties (i)-(iii). $\square$

Subdividing a conditional simplex $\mathcal{S} = \operatorname{conv}(X_1, \dots, X_N)$ barycentrically we obtain $(\mathcal{C}_\pi)_{\pi \in S_N}$. Dividing every $\mathcal{C}_\pi$ barycentrically results in a new collection of conditional simplexes and we call this the two-fold barycentric subdivision of $\mathcal{S}$. Inductively, we can subdivide every conditional simplex of the $(m-1)$th step barycentrically and call the resulting collection of conditional simplexes the m-fold barycentric subdivision of $\mathcal{S}$ and denote it by $\mathscr{S}^m$. Further, we define $\operatorname{ext}(\mathscr{S}^m) = \sigma(\{\operatorname{ext}(\mathcal{C}) \colon \mathcal{C} \in \mathscr{S}^m\})$ to be the σ-stable hull of all extremal points of the conditional simplexes of the m-fold barycentric subdivision of $\mathcal{S}$. Notice that this is the σ-stable hull of only finitely many elements, since there are only finitely many simplexes in the subdivision, each of which is the convex hull of N elements.

Remark 3.12. Consider an arbitrary $\mathcal{C}_\pi = \operatorname{conv}(Y_1^\pi, \dots, Y_N^\pi)$, $\pi \in \mathrm{S_N}$ in the barycentric subdivision of a conditional simplex $\mathcal{S}$. Then it holds that

$$\operatorname{diam}(\mathcal{C}_\pi) = \operatorname*{ess\,sup}_{i,j=1,\dots,N} \left\| Y_i^\pi - Y_j^\pi \right\| \leq \frac{N-1}{N} \operatorname{diam}(\mathcal{S}).$$

Since this holds for any $\pi \in S_N$, it follows that the diameter of $\mathcal{S}^m$, which is an arbitrary conditional simplex of the m-fold barycentric subdivision of $\mathcal{S}$, fulfills $\operatorname{diam}(\mathcal{S}^m) \leq \left(\frac{N-1}{N}\right)^m \operatorname{diam}(\mathcal{S})$. Since $\operatorname{diam}(\mathcal{S}) < \infty$ and $\left(\frac{N-1}{N}\right)^m \to 0$, for $m \to \infty$, it follows that $\operatorname{diam}(\mathcal{S}^m) \to 0$, for $m \to \infty$ for every sequence $(\mathcal{S}^m)_{m \in \mathbb{N}}$.

3.2 Brouwer Fixed Point Theorem for Conditional Simplexes

Definition 3.13. Let $\mathcal{S} = \operatorname{conv}(X_1, \dots, X_N)$ be a conditional simplex, m-fold barycentrically subdivided in $\mathscr{S}^m$. A local function $\phi \colon \operatorname{ext}(\mathscr{S}^m) \to \{1, \dots, N\}(\mathcal{A})$ is called a labeling function of $\mathcal{S}$. For fixed $X_1, \dots, X_N \in \operatorname{ext}(\mathcal{S})$ with $\mathcal{S} = \operatorname{conv}(X_1, \dots, X_N)$, the labeling function is called proper if for any $Y \in \operatorname{ext}(\mathscr{S}^m)$ it holds that

$$P\left(\{\omega\colon \phi(Y)(\omega) = i, \lambda_i(\omega) = 0\}\right) = 0,$$

for $i = 1, \dots, N$, where $Y = \sum_{i=1}^N \lambda_i X_i$. A conditional simplex $\mathcal{C} = \operatorname{conv}(Y_1, \dots, Y_N) \subseteq \mathcal{S}$, with $Y_j \in \operatorname{ext}(\mathscr{S}^m), j = 1, \dots, N$, is said to be completely labeled by ϕ if ϕ is a proper labeling function of $\mathcal{S}$ and

$$P\left(\{\omega\colon \text{ there exists } j \in \{1, \dots, N\}, \phi(Y_j)(\omega) = i\}\right) = 1$$

for all $i \in \{1, \dots, N\}$.

Lemma 3.14. *Let $\mathcal{S} = \operatorname{conv}(X_1, \dots, X_N)$ be a conditional simplex and $f\colon \mathcal{S} \to \mathcal{S}$ be a local function. Let $\phi\colon \operatorname{ext}(\mathscr{S}^m) \to \{0, \dots, N\}(\mathcal{A})$ be a local function such that for every $X \in \operatorname{ext}(\mathscr{S}^m)$ it holds that*

(i) $P\left(\{\omega\colon \phi(X)(\omega) = i; \lambda_i(\omega) = 0 \text{ or } \mu_i(\omega) > \lambda_i(\omega)\}\right) = 0$ *for all* $i = 1, \dots, N$,

(ii) $P\left(\{\omega\colon \phi(X)(\omega) = 0, \text{ there exists } i \in \{1, \dots, N\}, \lambda_i(\omega) > 0, \lambda_i(\omega) \geq \mu_i(\omega)\}\right) = 0,$

where $(\lambda_i)_{i=1,\dots,N}$ and $(\mu_i)_{i=1,\dots,N}$ are determined by $X = \sum_{i=1}^N \lambda_i X_i$ and $f(X) = \sum_{i=1}^N \mu_i X_i$. Then ϕ is a proper labeling function.

Moreover, the set of functions fulfilling these properties is nonempty.

Proof. First we show that ϕ is a labeling function. Since ϕ is local, we just have to prove that ϕ actually maps into $\{1, \dots, N\}$. Due to (ii), we have to show that

$$P(\{\omega\colon \text{ there exists } i \in \{1, \dots, N\}, \lambda_i(\omega) \geq \mu_i(\omega), \lambda_i(\omega) > 0\}) = 1.$$

Assume, to the contrary, that $\mu_i > \lambda_i$ on $A \in \mathcal{A}_+$ for all λ_i with $\lambda_i > 0$ on A. Then it holds that $1 = \sum_{i=1}^N \lambda_i 1_{\{\lambda_i > 0\}} < \sum_{i=1}^N \mu_i 1_{\{\mu_i > 0\}} = 1$ on A, which yields a contradiction. Thus, ϕ is a labeling function. Moreover, due to (i), it holds in particular that $P\left(\{\omega\colon \phi(X)(\omega) = i, \lambda_i(\omega) = 0\}\right) = 0$, which shows that ϕ is proper.

To prove the existence for $X \in \operatorname{ext}(\mathscr{S}^m)$ with $X = \sum_{i=1}^N \lambda_i X_i$, $f(X) = \sum_{i=1}^N \mu_i$, let $B_i := \{\omega\colon \lambda_i(\omega) > 0\} \cap \{\omega\colon \lambda_i(\omega) \geq \mu_i(\omega)\}$, $i = 1, \dots, N$. Then we define the function ϕ at X as $\{\omega\colon \phi(X)(\omega) = i\} = B_i \setminus (\bigcup_{k=1}^{i-1} B_k)$, $i = 1, \dots, N$. It has been shown that ϕ maps to $\{1, \dots, N\}(\mathcal{A})$ and is proper. It remains to show that ϕ is local. To this end, consider $X = \sum_{j \in \mathbb{N}} 1_{A_j} X^j$, where $X^j = \sum_{i=1}^N \lambda_i^j X_i$ and $f(X^j) = \sum_{i=1}^N \mu_i^j X_i$. Due to uniqueness of the coefficients in a conditional simplex. it holds that $\lambda_i = \sum_{j \in \mathbb{N}} 1_{A_j} \lambda_i^j$, and due to locality of f, it follows that $\mu_i = \sum_{j \in \mathbb{N}} 1_{A_j} \mu_i^j$. Therefore it holds that $B_i = \bigcup_{j \in \mathbb{N}} \left(\{\omega\colon \lambda_i^j(\omega) > 0\} \cap \{\omega\colon \lambda_i^j(\omega) \geq \mu_i^j(\omega)\} \cap A_j\right) = \bigcup_{j \in \mathbb{N}} (B_i^j \cap A_j)$. Hence, $\phi(X) = i$ on $B_i \setminus (\bigcup_{k=1}^{i-1} B_k) = [\bigcup_{j \in \mathbb{N}} (B_i^j \cap A_j)] \setminus [\bigcup_{k=1}^{i-1} (\bigcup_{j \in \mathbb{N}} B_k^j \cap A_j)] = \bigcup_{j \in \mathbb{N}} [(B_i^j \setminus \bigcup_{k=1}^{i-1} B_k^j) \cap A_j]$.

Reversely, we see that $\sum_{j\in\mathbb{N}} 1_{A_j}\phi(X^j)$ is i on any $A_j \cap \{\omega\colon \phi(X^j)(\omega) = i\}$, hence it is i on $\bigcup_{j\in\mathbb{N}}(B_i^j \setminus \bigcup_{k=1}^{i-1} B_k^j) \cap A_j$. Thus, $\sum_{j\in\mathbb{N}} 1_{A_j}\phi(X^j) = \phi(\sum_{j\in\mathbb{N}} 1_{A_j} X^j)$, which shows that ϕ is local. $\square$

The reason to demand locality of a labeling function is exactly because we want to label by the function ϕ mentioned in the existence proof of Lemma 3.14 and hence keep local information with it. For example, consider a conditional simplex $\mathcal{S} = \operatorname{conv}(X_1, X_2, X_3, X_4)$ and $\Omega = \{\omega_1, \omega_2\}$. Let $Y \in \operatorname{ext}(\mathscr{S})$ be given by $Y = \frac{1}{3}\sum_{i=1}^{3} X_i$. Now consider a function f on $\mathcal{S}$ such that

$$f(Y)(\omega_1) = \frac{1}{4}X_1(\omega_1) + \frac{3}{4}X_3(\omega_1); \quad f(Y)(\omega_2) = \frac{2}{5}X_1(\omega_2) + \frac{2}{5}X_2(\omega_2) + \frac{1}{5}X_4(\omega_2).$$

If we label Y by the rule explained in Lemma 3.14, ϕ takes the values $\phi(Y)(\omega_1) \in \{1, 2\}$ and $\phi(Y)(\omega_2) = 3$. Therefore, we can really distinguish on which sets $\lambda_i \geq \mu_i$. Yet, using a deterministic labeling of Y, we would lose this information.

Theorem 3.15. *Let $\mathcal{S} = \operatorname{conv}(X_1, \dots, X_N)$ be a conditional simplex in $(L^0)^d$. Let $f\colon \mathcal{S} \to \mathcal{S}$ be a local, sequentially continuous function. Then there exists $Y \in \mathcal{S}$ such that $f(Y) = Y$.*

Proof. We consider the barycentric subdivision $(\mathcal{C}_\pi)_{\pi\in \mathrm{S_N}}$ of $\mathcal{S}$ and a proper labeling function ϕ on $\operatorname{ext}(\mathscr{S})$. First, we show that we can find a completely labeled conditional simplex in $\mathcal{S}$. By induction on the dimension of $\mathcal{S} = \operatorname{conv}(X_1, \dots, X_N)$, we show that there exists a partition $(A_k)_{k=1,\dots,K}$ such that on any A_k there is an odd number of completely labeled $\mathcal{C}_\pi$. The case $N = 1$ is clear since a point can be labeled with the constant index 1 only.

Suppose that the case $N-1$ is proven. Since the number of Y_i^π of the barycentric subdivision is finite and ϕ can only take finitely many values, it holds for all $V \in (Y_i^\pi)_{i=1,\dots,N,\pi\in\mathrm{S_N}}$ that there exists a partition $(A_k^V)_{k=1,\dots,K}$, $K < \infty$, where $\phi(V)$ is constant on any A_k^V. Therefore, we find a partition $(A_k)_{k=1,\dots,K}$ such that $\phi(V)$ on A_k is constant for all V and A_k. Fix A_k now.

In the following, we denote by $\mathcal{C}_{\pi^b}$ those conditional simplexes for which $\mathcal{C}_{\pi^b} \cap \mathcal{B}_{N-1}$ are $N-1$-dimensional (compare Lemma 3.11 (iv)), therefore $\pi^b(N) = N$. Further we denote by $\mathcal{C}_{\pi^c}$ these conditional simplexes which are not of the type $\mathcal{C}_{\pi^b}$, that is, $\pi^c(N) \neq N$. If we use $\mathcal{C}_\pi$, we mean a conditional simplex of arbitrary type. We define

- $\mathscr{C} \subseteq (\mathcal{C}_\pi)_{\pi\in\mathrm{S_N}}$ to be the set of $\mathcal{C}_\pi$ which are completely labeled on A_k.
- $\mathscr{A} \subseteq (\mathcal{C}_\pi)_{\pi\in\mathrm{S_N}}$ to be the set of P-almost completely labeled $\mathcal{C}_\pi$, that is

$$\{\phi(Y_k^\pi), k \in \{1, \dots, N\}\} = \{1, \dots, N-1\} \quad \text{on } A_k.$$

- $\mathscr{E}_\pi$ to be the set of intersections $(\mathcal{C}_\pi \cap \mathcal{C}_{\pi_l})_{\pi_l \in \mathrm{S_N}}$ which are $N-1$-dimensional and completely labeled on A_k.[2]
- $\mathscr{B}_\pi$ to be the set of intersections $\mathcal{C}_\pi \cap \mathcal{B}_{N-1}$ which are completely labeled on A_k.

It holds that $\mathscr{E}_\pi \cap \mathscr{B}_\pi = \emptyset$ and hence $|\mathscr{E}_\pi \cup \mathscr{B}_\pi| = |\mathscr{E}_\pi| + |\mathscr{B}_\pi|$. Since $\mathcal{C}_{\pi^c} \cap \mathcal{B}_{N-1}$ is at most $N-2$-dimensional, it holds that $\mathscr{B}_{\pi^c} = \emptyset$ and hence $|\mathscr{B}_{\pi^c}| = 0$. Moreover, we know that $\mathcal{C}_\pi \cap \mathcal{C}_{\pi_l}$ is $N-1$-dimensional on A_k if and only if this holds on the whole Ω (compare Lemma 3.11 (iii)) and $\mathcal{C}_{\pi^b} \cap \mathcal{B}_{N-1} \neq \emptyset$ on A_k if and only if this also holds on the whole Ω (compare Lemma 3.11 (iv)). So, it does not play any role if we look at these sets which are intersections on A_k or on Ω since they are exactly the same sets.

If $\mathcal{C}_{\pi^c} \in \mathscr{C}$, then $|\mathscr{E}_{\pi^c}| = 1$ and if $\mathcal{C}_{\pi^b} \in \mathscr{C}$ then $|\mathscr{E}_{\pi^b} \cup \mathscr{B}_{\pi^b}| = 1$. If $\mathcal{C}_{\pi^c} \in \mathscr{A}$, then $|\mathscr{E}_{\pi^c}| = 2$ and if $\mathcal{C}_{\pi^b} \in \mathscr{A}$ then $|\mathscr{E}_{\pi^b} \cup \mathscr{B}_{\pi^b}| = 2$. Therefore it holds that $\sum_{\pi \in \mathrm{S_N}} |\mathscr{E}_\pi \cup \mathscr{B}_\pi| = |\mathscr{C}| + 2\,|\mathscr{A}|$.

If we pick $E_\pi \in \mathscr{E}_\pi$ we know that there always exists exactly one other π_l such that $E_\pi \in \mathscr{E}_{\pi_l}$ (Lemma 3.11(iii)). Therefore $\sum_{\pi \in \mathrm{S_N}} |\mathscr{E}_\pi|$ is even. Moreover, $(\mathcal{C}_{\pi^b} \cap \mathcal{B}_{N-1})_{\pi^b}$ subdivides $\mathcal{B}_{N-1}$ barycentrically, and hence we can apply the hypothesis (on $\mathrm{ext}(\mathcal{C}_{\pi^b} \cap \mathcal{B}_{N-1})$). Indeed, the set $\mathcal{B}_{N-1}$ is a σ-stable set, so if it is partitioned by the labeling function into $(A_k)_{k=1,\dots,K}$, we know that $\mathcal{B}_{N-1}(\mathcal{S}) = \sum_{k=1}^K 1_{A_k} \mathcal{B}_{N-1}(1_{A_k}\mathcal{S})$ and by Lemma 3.11 (iv) we can apply the induction hypothesis also to every $A_k, k = 1, \dots, K$. Thus, the number of completely labeled conditional simplexes is odd on a partition of Ω, but since ϕ is constant on A_k, it also has to be odd there. This means that $\sum_{\pi^b} |\mathscr{B}_{\pi^b}|$ has to be odd. Hence, we also have that $\sum_\pi |\mathscr{E}_\pi \cup \mathscr{B}_\pi|$ is the sum of an even and an odd number and thus odd. So, we conclude $|\mathscr{C}| + 2\,|\mathscr{A}|$ is odd and hence also $|\mathscr{C}|$. Thus, we find for any A_k a completely labeled $\mathcal{C}_{\pi_k}$.

We define $\mathcal{S}^1 = \sum_{k=1}^K 1_{A_k} \mathcal{C}_{\pi_k}$ which by Remark 3.9 is indeed a conditional simplex. Due to σ-stability of $\mathcal{S}$ it holds that $\mathcal{S}^1 \subseteq \mathcal{S}$. By Remark 3.12 $\mathcal{S}^1$ has a diameter which is less than $\frac{N-1}{N}\,\mathrm{diam}(\mathcal{S})$ and since ϕ is local, $\mathcal{S}^1$ is completely labeled on the whole Ω.

The same argumentation holds for every m-fold barycentric subdivision $\mathscr{S}^m$ of $\mathcal{S}$, $m \in \mathbb{N}$, that is, there exists a completely labeled conditional simplex in every m-fold barycentrically subdivided conditional simplex which is properly labeled. Henceforth, subdividing $\mathcal{S}$ m-fold barycentrically and labeling it by $\phi^m \colon \mathrm{ext}(\mathscr{S}^m) \to \{1, \dots, N\}(\mathcal{A})$, which is a labeling function as in Lemma 3.14, we always obtain a completely labeled conditional simplex $\mathcal{S}^{m+1} \subseteq \mathcal{S}$ for $m \in \mathbb{N}$. Moreover, since $\mathcal{S}^1$ is completely labeled, it holds that $\mathcal{S}^1 = \sum_{k=1}^K 1_{A_k} \mathcal{C}_{\pi_k}$ as above, where $\mathcal{C}_{\pi_k}$ is completely labeled on A_k. This means $\mathcal{C}_{\pi_k} = \mathrm{conv}(Y_1^k, \dots, Y_N^k)$ with $\phi(Y_j^k) = j$ on A_k for every $j = 1, \dots, N$. Defining $V_j^1 = \sum_{k=1}^K 1_{A_k} Y_j^k$ for every $j = 1, \dots, N$ yields $P(\{\omega : \phi(V_j^1)(\omega) = j\}) = 1$ for every $j = 1, \dots, N$ and $\mathcal{S}^1 = \mathrm{conv}(V_1^1, \dots, V_N^1)$. The same holds for any $m \in \mathbb{N}$ and so that we can write $\mathcal{S}^m = \mathrm{conv}(V_1^m, \dots, V_N^m)$ with $P(\{\omega \colon \phi^{m-1}(V_j^m)(\omega) = j\}) = 1$ for every $j = 1, \dots, N$.

[2] That is bearing exactly the label $1, \dots, N-1$ on A_k.

Now, $(V_1^m)_{m\in\mathbb{N}}$ is a sequence in the sequentially closed, L^0-bounded set $\mathcal{S}$, so that by [24, Corollary 3.9], there exists $Y \in \mathcal{S}$ and a sequence $(M_m)_{m\in\mathbb{N}}$ in $\mathbb{N}(\mathcal{A})$ such that $M_{m+1} > M_m$ for all $m \in \mathbb{N}$ and $\lim_{m\to\infty} V_1^{M_m} = Y$ P-almost surely. For $M_m \in \mathbb{N}(\mathcal{A})$, $V_1^{M_m}$ is defined as $\sum_{n\in\mathbb{N}} 1_{\{M_m=n\}} V_1^n$. This means an element with index M_m, for some $m \in \mathbb{N}$, equals V_1^n on A_n, $n \in \mathbb{N}$, where the sets A_n are determined by M_m via $A_n = \{\omega\colon M_m(\omega) = n\}$, $n \in \mathbb{N}$. Furthermore, as m goes to ∞, $\operatorname{diam}(S^m)$ is converging to zero P-almost surely, and therefore it also follows that $\lim_{m\to\infty} V_k^{M_m} = Y$ P-almost surely for every $k = 1, \dots, N$. Indeed, it holds that $|V_k^m - Y| \le diam(\mathcal{S}^m) + |V_1^m - Y|$ for every $k = 1, \dots, N$ and $m \in \mathbb{N}$, so we can use the sequence $(M_m)_{m\in\mathbb{N}}$ for every $k = 1, \dots, N$.

Let $Y = \sum_{l=1}^N \alpha_l X_l$ and $f(Y) = \sum_{l=1}^N \beta_l X_l$ as well as $V_k^m = \sum_{l=1}^N \lambda_l^{m,k} X_l$ and $f(V_k^m) = \sum_{l=1}^N \mu_l^{m,k} X_l$ for $m \in \mathbb{N}$. It holds that $f(V_1^{M_m}) = \sum_{n\in\mathbb{N}} 1_{\{M_m=n\}} f(V_1^n)$, since f is local,. By sequential continuity of f, it follows that $\lim_{m\to\infty} f(V_k^{M_n}) = f(Y)$ P-almost surely for every $k = 1, \dots, N$. In particular, $\lim_{m\to\infty} \lambda_l^{M_m,l} = \alpha_l$ and $\lim_{m\to\infty} \mu_l^{M_m,l} = \beta_l$ P-almost surely for every $l = 1, \dots, N$. However, by construction, $\phi^{m-1}(V_l^m) = l$ for every $l = 1, \dots, N$, and from the choice of ϕ^{m-1}, it follows that $\lambda_l^{m,l} \ge \mu_l^{m,l}$ P-almost surely for every $l = 1, \dots, N$ and $m \in \mathbb{N}$. Hence, $\alpha_l = \lim_{m\to\infty} \lambda_l^{M_m,l} \ge \lim_{m\to\infty} \mu_l^{M_m,l} = \beta_l$ P-almost surely for every $l = 1, \dots, N$. This is possible only if $\alpha_l = \beta_l$ P-almost surely for every $l = 1, \dots, N$, showing that $f(Y) = Y$. □

3.3 Applications

3.3.1 Fixed Point Theorem for Sequentially Closed and Bounded Sets in $(L^0)^d$

Proposition 3.16. *Let $\mathcal{K}$ be an L^0-convex, sequentially closed and bounded subset of $(L^0)^d$, and let $f\colon \mathcal{K} \to \mathcal{K}$ be a local, sequentially continuous function. Then f has a fixed point.*

Proof. Since $\mathcal{K}$ is bounded, there exists a conditional simplex $\mathcal{S}$ such that $\mathcal{K} \subseteq \mathcal{S}$. Now define the function $h\colon \mathcal{S} \to \mathcal{K}$ by

$$h(X) = \begin{cases} X, & \text{if } X \in \mathcal{K}, \\ \arg\min\{\|X - Y\| : Y \in \mathcal{K}\}, & \text{else.} \end{cases}$$

This means, that h is the identity on $\mathcal{K}$ and the projection on $\mathcal{K}$ for the elements in $\mathcal{S} \setminus \mathcal{K}$. Due to [24, Corollary 4.5] this minimum exists and is unique. Therefore h is well defined.

We can characterize h by

$$Y = h(X) \Leftrightarrow \langle X - Y, Z - Y\rangle \le 0 \ \text{ for all } Z \in \mathcal{K}. \tag{3.6}$$

Indeed, let $\langle X-Y, Z-Y\rangle \leq 0$ for all $Z \in \mathcal{K}$. Then

$$\begin{aligned}\|X-Z\|^2 &= \|(X-Y)+(Y-Z)\| \\ &= \|X-Y\|^2 + 2\langle X-Y, Y-Z\rangle + \|Y-Z\|^2 \geq \|X-Y\|^2,\end{aligned}$$

which shows the minimizing property of h. Reversely, let $Y = h(X)$. Since $\mathcal{K}$ is L^0-convex, $\lambda Z + (1-\lambda)Y \in \mathcal{K}$ for any $\lambda \in (0,1](\mathcal{A})$ and $Z \in \mathcal{K}$. By a standard calculation,

$$\|X - (\lambda Z + (1-\lambda)Y)\|^2 \geq \|X-Y\|^2$$

yields $0 \geq -2\lambda\langle X, -Y\rangle + (2\lambda - \lambda^2)\langle Y, Y\rangle + 2\lambda\langle X, Z\rangle - \lambda^2 \|Z\|^2 - 2\lambda(1-\lambda)\langle Z, Y\rangle$. Dividing by $\lambda > 0$ and letting $\lambda \downarrow 0$ afterwards yields

$$0 \geq -2\langle X, -Y\rangle + 2\langle Y, Y\rangle + 2\langle X, Z\rangle - 2\langle Z, Y\rangle = 2\langle X-Y, Z-Y\rangle,$$

which is the desired claim.

Furthermore, for any $X, Y \in \mathcal{S}$, it holds that

$$\|h(X) - h(Y)\| \leq \|X-Y\|.$$

Indeed,

$$X - Y = (h(X) - h(Y)) + X - h(X) + h(Y) - Y =: (h(X) - h(Y)) + c,$$

which means

$$\|X-Y\|^2 = \|h(X)-h(Y)\|^2 + \|c\|^2 + 2\langle c, h(X) - h(Y)\rangle. \tag{3.7}$$

Since

$$\langle c, h(X)-h(Y)\rangle = -\langle X - h(X), h(Y) - h(X)\rangle - \langle Y - h(Y), h(X) - h(Y)\rangle,$$

by (3.6), it follows that $\langle c, h(X) - h(Y)\rangle \geq 0$ and (3.7) implies that $\|X-Y\|^2 \geq \|h(X)-h(Y)\|^2$. This shows that h is sequentially continuous.

The function $f \circ h$ is a sequentially continuous function mapping from $\mathcal{S}$ to $\mathcal{K} \subseteq \mathcal{S}$. Hence, there exists a fixed point $f \circ h(Z) = Z$. Since $f \circ h$ maps into $\mathcal{K}$, this Z has to be in $\mathcal{K}$. But then we know $h(Z) = Z$ and therefore $f(Z) = Z$, which ends the proof. □

Remark 3.17. In Drapeau et al. [31] the concept of conditional compactness is introduced and it is shown that there is an equivalence between conditional compactness and conditional closed- and boundedness in $(L^0)^d$. In that context we can formulate the conditional Brouwer fixed point theorem as follows. A sequentially continuous function $f\colon \mathcal{K} \to \mathcal{K}$ such that $\mathcal{K}$ is a conditionally compact and L^0-convex subset of $(L^0)^d$ has a fixed point.

3.3.2 Applications in Conditional Analysis on $(L^0)^d$

Working in $\mathbb{R}^d$, the Brouwer fixed point theorem can be used to prove several topological properties and is even equivalent to some of them. In the theory of $(L^0)^d$, we will show that the conditional Brouwer fixed point theorem has several implications as well.

Define the unit ball in $(L^0)^d$ by $\mathcal{B}(d) = \{X \in (L^0)^d \colon \|X\| \le 1\}$. Then, by the former theorem, any local, sequentially continuous function $f\colon \mathcal{B}(d) \to \mathcal{B}(d)$ has a fixed point. The unit sphere $\mathcal{S}(d-1)$ is defined as $\mathcal{S}(d-1) = \{X \in (L^0)^d \colon \|X\| = 1\}$.

Definition 3.18. Let $\mathcal{X}$ and $\mathcal{Y}$ be subsets of $(L^0)^d$. An L^0-homotopy of two local, sequentially continuous functions $f, g\colon \mathcal{X} \to \mathcal{Y}$ is a jointly local, sequentially continuous function $H\colon \mathcal{X} \times [0,1](\mathcal{A}) \to \mathcal{Y}$ such that $H(X,0) = f(X)$ and $H(X,1) = g(X)$. Jointly local means $H(\sum_{j\in\mathbb{N}} 1_{A_j} X_j, \sum_{j\in\mathbb{N}} 1_{A_j} t_j) = \sum_{j\in\mathbb{N}} 1_{A_j} H(X_j, t_j)$ for any partition $(A_j)_{j\in\mathbb{N}}$, $(X_j)_{j\in\mathbb{N}}$ in $\mathcal{X}$ and $(t_j)_{j\in\mathbb{N}}$ in $[0,1](\mathcal{A})$. Sequential continuity of H is therefore $H(X_n, t_n) \to H(X,t)$ whenever $X_n \to X$ and $t_n \to t$ both P-almost surely for $X_n, X \in \mathcal{X}$ and $t_n, t \in [0,1](\mathcal{A})$.

Lemma 3.19. *The identity function of the sphere is not L^0-homotopic to a constant function.*

The proof is a consequence of the following lemma.

Lemma 3.20. *There does not exist a local, sequentially continuous function $f\colon \mathcal{B}(d) \to \mathcal{S}(d-1)$ which is the identity on $\mathcal{S}(d-1)$.*

Proof. Suppose that there is this local, sequentially continuous function f. Define $g\colon \mathcal{S}(d-1) \to \mathcal{S}(d-1)$ by $g(X) = -X$. Then the composition $g \circ f\colon \mathcal{B}(d) \to \mathcal{B}(d)$, which actually maps to $\mathcal{S}(d-1)$, is local and sequentially continuous. Therefore, this has a fixed point Y which has to be in $\mathcal{S}(d-1)$ since this is the image of $g \circ f$. But we know $f(Y) = Y$ and $g(Y) = -Y$ and hence $g \circ f(Y) = -Y$. Therefore, Y cannot be a fixed point (since $0 \notin \mathcal{S}(d-1)$) which is a contradiction. $\square$

It directly follows that the identity on the sphere is not L^0-homotopic to a constant function. In the case $d = 1$, we get the following result which is the L^0-module version of an intermediate value theorem.

Lemma 3.21. *Let $X, \overline{X} \in L^0$ with $X \le \overline{X}$. Let $\left[X, \overline{X}\right] = \left\{Z \in L^0 \colon X \le Z \le \overline{X}\right\}$ and $f\colon \left[X, \overline{X}\right] \to L^0$ be a local, sequentially continuous function. Define $A = \{f(X) \le f\left(\overline{X}\right)\}$. Then for every $Y \in \left[1_A f(X) + 1_{A^c} f\left(\overline{X}\right), 1_A f\left(\overline{X}\right) + 1_{A^c} f(X)\right]$ there exists $\overline{Y} \in \left[X, \overline{X}\right]$ with $f\left(\overline{Y}\right) = Y$.*

Proof. Since f is local, it is sufficient to prove the case for $f(X) \leq f\left(\overline{X}\right)$ which is $A = \Omega$. For the general case, we would consider A and A^c separately, obtain $1_A f\left(\overline{Y}_1\right) = 1_A Y$, $1_{A^c} f\left(\overline{Y}_2\right) = 1_{A^c} Y$ and by locality we have $f\left(1_A \overline{Y}_1 + 1_{A^c} \overline{Y}_2\right) = Y$. So, suppose $Y \in \left[f(X), f\left(\overline{X}\right)\right]$ in the rest of the proof.

Let first $f(X) < Y < f\left(\overline{X}\right)$. Define the function $g\colon \left[X, \overline{X}\right] \to \left[X, \overline{X}\right]$ by

$$g(V) := p(V - f(V) + Y) \quad \text{with} \quad p(Z) = 1_{\{Z \leq X\}} X + 1_{\{X \leq Z \leq \overline{X}\}} Z + 1_{\{\overline{X} \leq Z\}} \overline{X}.$$

Notice that as a sum, product, and composition of local, sequentially continuous functions, g is so as well. Hence, g has a fixed point $\overline{Y}$. If $\overline{Y} = X$, it must hold that $X - f(X) + Y \leq X$, which means $Y \leq f(X)$, which is a contradiction. If $\overline{Y} = \overline{X}$, it follows that $f\left(\overline{X}\right) \leq Y$, which is also a contradiction. Hence, $\overline{Y} = \overline{Y} - f\left(\overline{Y}\right) + Y$, which means $f\left(\overline{Y}\right) = Y$.

If $Y = f(X)$ on B and $Y = f\left(\overline{X}\right)$ on C, it holds that $f(X) < Y < f\left(\overline{X}\right)$ on $(B \cup C)^c =: D$. Then we find $\overline{Y}$ such that $f\left(\overline{Y}\right) = Y$ on D. In total

$$f\left(1_B X + 1_{C \setminus B} \overline{X} + 1_D \overline{Y}\right) = 1_B f(X) + 1_{C \setminus B} f\left(\overline{X}\right) + 1_D f\left(\overline{Y}\right) = Y.$$

This shows the claim for general $Y \in \left[f(X), f(\overline{X})\right]$. □

4 Conditional Topological Vector Spaces

In this chapter, we work with conditional sets, a concept introduced in [31]. Let us motivate the concept of conditional set theory by two examples. In L^0, consider stochastic intervals which are sets of the form $[X, Y] = \{Z \in L^0 : X \leq Z \leq Y\}$ where $X, Y \in L^0$ and $X \leq Y$. Working with the normal set operations, the union of $[0,1] \cup [1,2]$ does not contain random variables such as $1_A + 2\ 1_{A^c}$, where 1_A denotes the indicator function of some $A \in \mathcal{F}$ (with $0 < P(A) < 1$). Hence, the union does not contain elements which are partly in one set and partly in the other. Thus, it holds that $[0,1] \cup [1,2] \neq [0,2]$ and moreover, $[0,1] \cap [0.5, 2] \neq [0.5, 1]$. The question arises, if there are operations $\sqcap$ and $\sqcup$ such that $[0,1] \sqcup [1,2] = [0,2]$ and $[0,1] \sqcap [0.5,2] = [0.5,1]$.

In the previous chapter about the Brouwer Fixed Point Theorem in $(L^0)^d$, we considered conditional simplexes and worked with local functions f. We showed the fixed point theorem in particular for functions from a conditional simplex to itself. Taking countably many conditional simplexes $(S_n)_{n\in\mathbb{N}}$ the object $S = \sum_{n\in\mathbb{N}} 1_{A_n} S_n$ could not been handled with the language of L^0-theory. However, inspired by the concept of locality of a function f, it formally holds that $f(\sum_{n\in\mathbb{N}} 1_{A_n} S_n) = \sum_{n\in\mathbb{N}} 1_{A_n} f(S_n)$. Hence, if f has a fixed point X_n in S_n, which is $f(X_n) = X_n$, the element $X = \sum_{n\in\mathbb{N}} 1_{A_n} X_n$ fulfills $X = f(X)$ and is hence a fixed point in S. To obtain this easy consequence of the Brouwer Fixed Point Theorem in $(L^0)^d$ the only thing to do is to give a formal description of the object S. The language of conditional set theory is tailored for this purpose.

This chapter is organized as follows. In the first section, we give a summary of the concept of conditional sets and recall important results needed later on. The proof of these results can be found in [31]. In the second section, we attend to our actual objective, namely convex analysis on conditional topological vector spaces. We introduce the concepts of vector spaces and duality in a conditional set theoretical framework and define objects such as the norm or the polar cone for it. Subsequently, we prove theorems of functional analysis in this framework: Hahn-Banach, Banach-Alaoglu and Krein-Šmulian.

4.1 Introduction to Conditional Set Theory

Let $\mathcal{A} = (\mathcal{A}, \wedge, \vee, ^c, 0, 1)$ be a Boolean algebra. The relation $a \leq b$ if $a \wedge b = a$ defines a distributive complemented lattice. For any family $(a_i)_{i\in I}$ in $\mathcal{A}$, we denote by $\vee a_i = \vee_{i\in I} a_i$ and $\wedge a_i = \wedge_{i\in I} a_i$ its supremum and infimum, respectively. For $a \in \mathcal{A}$, the relative algebra of $\mathcal{A}$ with respect to a is denoted by $\mathcal{A}_a$. A partition of an element $a \in \mathcal{A}$ is a family $(a_i)_{i\in I}$ in $\mathcal{A}$ such that $\vee a_i = a$ and $a_i \wedge a_j = 0$ if $i \neq j$. We denote by $\mathcal{K}(a)$ the set of all partitions of a. Denote by $\mathscr{A}$ the class of all complete Boolean algebras satisfying the following:

(P) For every family $(a_i)_{i\in I}$ in $\mathcal{A}$ there exists a partition $(b_j)_{j\in J} \in \mathcal{K}(\vee a_i)$ such that for all $j \in J$ there is $i_j \in I$ with $b_j \leq a_{i_j}$.

The power set algebra $\mathcal{P}(X)$ of any set X is in $\mathscr{A}$ as well as the associated measure algebra of a σ-finite measure space , compare [63, Chapter 22].

Definition 4.1. Let $\mathcal{A} \in \mathscr{A}$, $(X_a)_{a\in\mathcal{A}}$ be a family of sets, and $(\gamma_a)_{a\in\mathcal{A}}$ be a family of surjective functions $\gamma_a : X_1 \to X_a$. The structure

$$X := (X_a, \gamma_a)_{a\in\mathcal{A}}$$

is a conditional set if and only if

(i) X_0 is a singleton;

(ii) (identity) γ_1 is the identity;

(iii) (consistency) $\gamma_a(x) = \gamma_a(y)$, whenever $\gamma_b(x) = \gamma_b(y)$, $x, y \in X_1$ and $a \leq b$;

(iv) ($\mathcal{A}$-stability) for every partition of unity $(a_i)_{i\in I}$ and for every $(x_i)_{i\in I} \in \prod_{i\in I} X_{a_i}$ there exists a unique $x \in X_1$ such that $\gamma_{a_i}(x) = x_i$ for all $i \in I$.

We identify two conditional sets X and Y if the only difference is $X_0 \neq Y_0$. Therefore, there exists only one conditional set on the degenerate algebra $\mathcal{A} = \{0\}$ which is denoted by $\mathbf{0}$.

We show how to generate a conditional set from a nonempty set E with respect to some $\mathcal{A} \in \mathscr{A}$. Denote by $\sum_{i\in I} a_i x_i := (a_i, x_i)_{i\in I}$ for every $(a_i, x_i)_{i\in I} \subseteq \mathcal{A} \times E$ for which $(a_i) \in \mathcal{K}(a)$ for some $a \in \mathcal{A}$. Define

$$E_a := \left\{ \sum a_i x_i : (a_i, x_i)_{i\in I} \subseteq \mathcal{A} \times E \right\},$$

where two families $\sum a_i x_i$ and $\sum b_j y_j$ are identified if $\vee\{a_i : x_i = z\} = \vee\{b_j : y_j = z\}$ for all $z \in E$. Let $\gamma_a : E_1 \to E_a$ be given by $\sum a_i x_i \mapsto \sum (a \wedge a_i) x_i$. In this way $(E_a, \gamma_a)_{a\in\mathcal{A}}$ is a conditional set.

Definition 4.2. Let $\mathcal{A} \in \mathscr{A}$ and E be a nonempty set. Then define $\mathbf{E} := (E_a, \gamma_a)_{a\in\mathcal{A}}$. For $E = \mathbb{N}$, $\mathbb{Z}$ or $\mathbb{Q}$, we call $\mathbf{N}$, $\mathbf{Z}$ and $\mathbf{Q}$ the conditional natural numbers, integers, and rational numbers with respect to $\mathcal{A}$, respectively.

Definition 4.3. Let $X = (X_a, \gamma_a)_{a\in\mathcal{A}}$ be a conditional set and $a \in \mathcal{A}$. A nonempty subset $Y \subseteq X_a$ is $\mathcal{A}_a$-stable if $\sum a_i x_i \in Y$ for all $[a_i, x_i] \subseteq \mathcal{A}_a \times X$ such that $x_i \in Y$ for all $i \in I$. For each $\mathcal{A}_a$-stable subset Y, a conditional set $Y := (Y_b, \delta_b)_{b\in\mathcal{A}_a}$ is associated, where $Y_b := \gamma^a_b(Y)$ and δ_b is the restriction of γ^a_b to Y_b for every $b \in \mathcal{A}_a$. In this case we say that Y lives on a.

For any nonempty $Y \subseteq X_a$, the set

$$\left\{\sum a_i x_i : [a_i, x_i] \subseteq \mathcal{A}_a \times X, (x_i) \subseteq Y\right\}$$

is an $\mathcal{A}_a$-stable subset of X_a. The associated conditional set is called the $\mathcal{A}$-stable hull of Y, and is denoted by $\operatorname{cond}(Y)$. Let $X = (X_a, \gamma_a)_{a\in\mathcal{A}}$ be a conditional set. Then the set X_a is $\mathcal{A}_a$-stable for each $a \in \mathcal{A}$. The associated conditional set is called the restriction of X to a, and is denoted by aX.

Definition 4.4. Let $X = (X_a, \gamma_a)_{a\in\mathcal{A}}$ and $Y = (Y_b, \delta_b)_{b\in\mathcal{B}}$ be two conditional sets. We say that Y is conditionally included in X, and write $Y \sqsubseteq X$ if and only if $\mathcal{B} = \mathcal{A}_a$ for some $a \in \mathcal{A}$, $Y_a \subseteq X_a$ is $\mathcal{A}_a$-stable, and Y is the conditional set associated to Y_a.

Proposition 4.5. *Let $X = (X_a, \gamma_a)_{a\in\mathcal{A}}$ be a conditional set. Then*

$$\mathcal{S}(X) = (\mathcal{S}_a, \delta_a)_{a\in\mathcal{A}}$$

is a conditional set, where $\mathcal{S}_a$ is the collection of all conditional sets Y associated to some $\mathcal{A}_a$-stable subset of X_a and $\delta_a : \mathcal{S}_1 \to \mathcal{S}_a$ is given by $Y \mapsto aY$, $a \in \mathcal{A}$. It holds that $a\mathcal{S}(X) = \mathcal{S}(aX)$.

Furthermore,

$$\mathcal{P}(X) = (\mathcal{P}_a, \delta_a)_{a\in\mathcal{A}}$$

is a conditional set where $\mathcal{P}_a := \{bY : Y \in \mathcal{S}(aX), b \in \mathcal{A}_a\}$ and $\delta_a : \mathcal{P}_1 \to \mathcal{P}_a$ is given by $bY \mapsto (a \wedge b)Y$ for each $a \in \mathcal{A}$. Moreover, $a\mathcal{P}(X) = \mathcal{P}(aX)$ and $\mathcal{S}(X) \sqsubseteq \mathcal{P}(X)$.

Theorem 4.6. *The structure $(\mathcal{P}(X), \sqcap, \sqcup, {}^{\sqsubset}, \mathbf{0}, X)$ is a complete Boolean algebra for every conditional set X, where*

(i) $\sqcup_{i\in I} Y^i$ is the conditional set associated to

$$\left\{\sum_{j\in J} b_j y_j : (b_j)_{j\in J} \in \mathcal{K}(\vee a_i), y_j \in Y^{i_j}, b_j \leq a_{i_j} \text{ for some } i_j \in I \text{ for all } j \in J\right\};$$

(ii) $\sqcap_{i\in I} Y^i$ *is the conditional set associated to*

$$\bigcap_{i\in I} a_* Y^i \quad \text{and} \quad a_* := \vee\left\{a\in\mathcal{A} : a \le \wedge a_i, \bigcap_{i\in I} aY^i \neq \emptyset\right\};$$

(iii) $Y^{\sqsubset} := \sqcup\{Z\in\mathcal{P}(X) : Y\sqcap Z = \mathbf{0}\}$

for $Y\in\mathcal{P}(X)$ *and* $(Y^i)_{i\in I}\subseteq\mathcal{P}(X)$, *where* Y^i *lives on* a_i *for every* $i\in I$.

Remark 4.7. Let $Y, Z\in\mathcal{S}(X)$ and $Y^1, Y^2\in\mathcal{P}(X)$ for some conditional set X. Then

(i) $Y\sqcap Z = Y\cap Z$, whenever $Y\sqcap Z\in\mathcal{S}(X)$;

(ii) $Y\sqsubseteq Z$ implies $Y\subseteq Z$;

(iii) $Y\sqcup Z = \mathrm{cond}(Y\cup Z)$;

(iv) $Y^1\sqcup Y^2 = b_1Y^1 + b_2(Y^1\sqcup Y^2) + b_3Y^2$, where $b_1 = a_1\wedge a_2^c$, $b_2 = a_1\wedge a_2$, $b_3 = a_2\wedge a_1^c$ and Y^i lives on a_i for each $i = 1, 2$.

4.1.1 Conditional Functions

For the reminder of this chapter, we fix a nondegenerate $\mathcal{A}\in\mathscr{A}$.

Definition 4.8. Let (X^i) be a family of conditional sets where X^i is a conditional set on $\mathcal{A}_a$ for some $a\in\mathcal{A}$ and each i. The conditional Cartesian product of the family (X^i) is

$$\prod X^i := \left(\prod X_b^i, (\gamma_b^i)\right)_{b\in\mathcal{A}_a},$$

if the index set is nonempty, and $\mathbf{0}$, otherwise.

Fix two conditional sets X and Y.

Definition 4.9. A conditional binary relation is a conditional set $R\sqsubseteq X\times Y$. For a pair (x, y) in $X\times Y$, we write xR_by if $(bx, by)\in R_b$ for $b\in\mathcal{A}$. Let $R\sqsubseteq X\times X$ be a conditional binary relation on X where R lives on a. We say that R is conditionally reflexive, symmetric, antisymmetric, or transitive if every R_b is reflexive, symmetric, antisymmetric, or transitive, respectively, for all $b\le a$.

Definition 4.10. A conditional relation $R\sqsubseteq X\times X$ is a conditional partial order or equivalence relation on X if R_1 is a partial order or an equivalence relation on X_1. Let $(X, \preccurlyeq)$ be a conditionally partially ordered set. We say that $Y\in\mathcal{S}(X)$ has a conditional upper bound, lower bound, supremum and infimum if Y_1 does so with respect to $\preccurlyeq_1$ in X_1. Moreover, Y is said to be conditionally bounded if it has a conditional upper and lower bound in X.

If $(X, \leqslant)$ is a conditionally partially ordered set, then we define $x < y$ whenever $x \leqslant y$ and $ax =_a ay$ implies $a = 0$. Moreover, for $x, y \in X$ with $x \leqslant y$ the following subsets of X_1 are $\mathcal{A}$-stable:

$$\{z \leqslant y\} := \{z \in X : z \leqslant y\}, \qquad \{x \leqslant z \leqslant y\} := \{z \in X : x \leqslant z \leqslant y\},$$
$$\{z < x\} := \{z \in X : z < x\}, \qquad \{x < z < y\} := \{z \in X : x < z < y\}.$$

Definition 4.11. Let X and Y be two conditional sets. A conditional function f from X into Y is a conditional binary relation $G_f \sqsubseteq X \times Y$ where G_{f_a} is the graph of a function $f_a : X_a \to Y_a$ for every $a \in \mathcal{A}$. We denote a conditional function by $f : X \to Y$ and, if there is no risk of confusion, we identify f with f_1.

A conditional family $(x_j) = (x_j)_{j \in J}$ is an element $x \in \mathcal{M}(J, X)$. A family $(x_j)_{j \in J} \subseteq X$ is in $\mathcal{M}(J, X)$ if and only if $\{x_j : j \in J\}$ is in $\mathcal{S}(X)$, since both is equivalent to $\sum a_i x_{j_i} = x_{\sum a_i j_i}$. Thus, $\sqcup x_j = \cup x_j$, due to Remark 4.7.

Definition 4.12. Let $f : X \to Y$ be a conditional function, $U \in \mathcal{S}(aX)$ and $V \in \mathcal{S}(bY)$. The conditional image of U is the conditional set associated to $f(U) := \{f_a(x) : x \in U\}$ and the conditional preimage of V is the one associated to $f^{-1}(V) := \{x \in b_* X : f_{b_*}(x) \in b_* V\}$ where $b_* = \vee\{c \leq b : f_c^{-1}(V_c) \neq \emptyset\}$.

Proposition 4.13. *Let $f : X \to Y$ be a conditional function, $[a_i, U^i] \subseteq \mathcal{A} \times \mathcal{P}(X)$ and $[a_i, V^i] \subseteq \mathcal{A} \times \mathcal{P}(Y)$, $(U^j) \subseteq \mathcal{P}(X)$ and $(V^j) \subseteq \mathcal{P}(Y)$, $U \in \mathcal{P}(X)$ and $V \in \mathcal{P}(Y)$, $U^1, U^2 \in \mathcal{P}(X)$ such that $U^1 \sqsubseteq U^2$ and $V^1, V^2 \in \mathcal{P}(Y)$ such that $V^1 \sqsubseteq V^2$. Then it holds*

$$\begin{aligned}
f\Big(\sum a_i U^i\Big) &= \sum a_i f(U^i) & f^{-1}\Big(\sum a_i V^i\Big) &= \sum a_i f^{-1}(V^i) \\
f(\sqcup U^j) &= \sqcup f(U^j) & f^{-1}(\sqcup V^j) &= \sqcup f^{-1}(V^j) \\
f(\sqcap U^j) &\sqsubseteq \sqcap f(U^j) & f^{-1}(\sqcap V^j) &= \sqcap f^{-1}(V^j) \\
f(U)^{\sqsubset} \sqcap f(X) &\sqsubseteq f(U^{\sqsubset}) & f^{-1}(V^{\sqsubset}) &= f^{-1}(V)^{\sqsubset} \\
f(U^1) &\sqsubseteq f(U^2) & f^{-1}(V^1) &\sqsubseteq f^{-1}(V^2) \\
U &\sqsubseteq f^{-1}(f(U)) & f(f^{-1}(V)) &\sqsubseteq V
\end{aligned} \tag{4.1}$$

and it is even an equality on the left-hand side of (4.1) *if f is conditionally injective (that is any $f_a, a \in \mathcal{A}$, is injective) , and on the right-hand side if $V \sqsubseteq f(X)$.*

4.1.2 Conditional Topology and Compactness

Throughout this section we fix a conditional set X.

Definition 4.14. A conditional topology on X is a family $\mathcal{T}$ in $\mathcal{P}(\mathcal{P}(X))$ satisfying the following properties:

(i) $\mathbf{0}, X \in \mathcal{T}$;

(ii) $\mathcal{T}$ is closed under finite conditional intersections;

(iii) $\mathcal{T}$ is closed under arbitrary conditional unions.

The elements of $\mathcal{T}$ are called conditionally open sets and their conditional complements conditionally closed sets. A conditional topological space $X = (X, \mathcal{T})$ is X endowed with a conditional topology $\mathcal{T}$. Given two conditional topologies $\mathcal{T}^1$ and $\mathcal{T}^2$ on X, we say that $\mathcal{T}^1$ is conditionally weaker than $\mathcal{T}^2$ if $\mathcal{T}^1 \sqsubseteq \mathcal{T}^2$.

Given $O \in \mathcal{T}$ and $a \in \mathcal{A}$, it holds $aO = aO + a^c\mathbf{0} \in \mathcal{T}$. Therefore, $a\mathcal{T}$ is a conditional topology on aX for every $a \in \mathcal{A}$. The conditional intersection of any family $(\mathcal{T}^i)$ of conditional topologies on X is itself a conditional topology on X. Indeed, since $X \in \mathcal{T}^i$ for all i, the conditional intersection of the family $(\mathcal{T}^i)$ coincides with their intersection.

Definition 4.15. The conditional topology generated by some conditional set $\mathcal{G}$ in $\mathcal{P}(\mathcal{P}(X))$ is defined as

$$\mathcal{T}^{\mathcal{G}} = \bigsqcap \{\mathcal{T} : \mathcal{G} \sqsubseteq \mathcal{T}, \mathcal{T} \text{ conditional topology on } X\}.$$

Due to $\sqcap_{\mathbf{0}} = X$, distributivity and associativity, it holds

$$\mathcal{T}^{\mathcal{G}} := \left\{ \bigsqcup_{i \in I} \bigsqcap_{j \in J_i} O^{ij} : O^{ij} \in \mathcal{G}, J_i \text{ conditionally finite, } I \text{ arbitrary} \right\}.$$

Definition 4.16. Let $(X, \mathcal{T})$ be a conditional topological space. A collection of sets $\mathcal{B} \sqsubseteq \mathcal{T}$ is a conditional topological base of $\mathcal{T}$ if $\mathcal{B} \in \mathcal{S}(\mathcal{S}(X))$ and for every $O \in \mathcal{T}$ there exist families (a_i) in $\mathcal{A}$ and (O_i) in $\mathcal{B}$ such that $O = \sqcup a_i O_i$.

Definition 4.17. Let $(X, \mathcal{T})$ be a conditional topological space and $Y \in \mathcal{S}(X)$. A conditional set $U \sqsubseteq X$ is a conditional neighborhood of Y if there exists $O \in \mathcal{T}$ such that $Y \sqsubseteq O \sqsubseteq U$. By $\mathcal{U}(Y)$ we denote the set of all conditional neighborhoods of Y. A conditional neighborhood base of an element $x \in X$ is a conditional set $\mathcal{V} \in \mathcal{S}(\mathcal{S}(X))$ such that for every $U \in \mathcal{U}(x)$ there exists $V \in \mathcal{V}$ with $x \in V \sqsubseteq U$. The conditional topological space X is conditionally Hausdorff if for every pair $x, y \in X$ with $x \sqcap y = \mathbf{0}$ there exists a pair of conditional neighborhoods U_x and U_y such that $U_x \sqcap U_y = \mathbf{0}$.

Definition 4.18. Let X be a conditional topological space and $Y \in \mathcal{S}(X)$. We define

$$\operatorname{int}(Y) = \{x \in a_* X : x \in U \sqsubseteq Y \text{ for some } U \in \mathcal{U}(x)\},$$
$$\operatorname{cl}(Y) = \{x \in X : U \sqcap Y \in \mathcal{S}(X) \text{ for all } U \in \mathcal{U}(x)\},$$

where $a_* = \vee\{a \in \mathcal{A} : O \sqsubseteq Y, O \text{ lives on } a\}$.

Definition 4.19. Let X and X' be conditional topological spaces. A conditional function $f : X \to Y$ is conditionally continuous at $x \in X$ if $f^{-1}(U)$ is a conditional neighborhood of x for every conditional neighborhood U of $f(x)$. A conditional function is said to be conditionally continuous on X if it is conditionally continuous at every $x \in X$.

Proposition 4.20. *Let $f : X \to X'$ be a conditional function between two conditional topological spaces. Then the following statements are equivalent:*

(i) *f is conditionally continuous;*

(ii) *$f^{-1}(O)$ is conditionally open in X for every conditionally open set O in X';*

(iii) *$f^{-1}(F)$ is conditionally closed in X for every conditionally closed set F in X';*

(iv) *$f^{-1}(\operatorname{int}(Z)) \sqsubseteq \operatorname{int}(f^{-1}(Z))$ for every $Z \sqsubseteq X'$;*

(v) *$f(\operatorname{cl}(Z)) \sqsubseteq \operatorname{cl}(f(Z))$ for every $Z \sqsubseteq X$.*

Definition 4.21. Let X be a conditional set, $(X_i, \mathcal{T}_i)$ be a family of conditional topological spaces and (f_i) be a family of conditional functions $f_i : X \to X_i$. The conditional topology $\mathcal{T}$ on X generated by $\operatorname{cond}(\mathcal{G})$ where

$$\mathcal{G} := \{f_i^{-1}(O_i) : O_i \in \mathcal{T}_i \text{ for some } i\},$$

is called the conditional initial topology on X for the family (f_i).

By construction, the conditional initial topology $\mathcal{T}$ on X for the family (f_i) is the conditionally weakest topology for which every f_i is conditionally continuous.

Definition 4.22. For a conditional direction $(J, \leqslant)$, we call the conditional family $(x_j)_{j\in J} \sqsubseteq X$ a conditional net. A conditional net $(y_\beta)_{\beta\in K}$ is called a conditional subnet of $(x_\alpha)_{\alpha\in J}$ if there exists a conditional function $\phi : K \to J$ such that $x_{\phi(\beta)} = y_\beta$, and for any $\alpha_0 \in J$ there exists $\beta_0 \in K$ such that $\beta \geqslant \beta_0$ implies $\phi(\beta) \geqslant \alpha_0$. If $(J, \leqslant) = (\mathbf{N}, \leqslant)$, we say that $(x_\alpha) = (x_n)$ is a conditional sequence.

Definition 4.23. Let X be a conditional topological space and $(x_\alpha) \subseteq X$ be a conditional net. An element $x \in X$ is a conditional

(i) limit point of (x_α) if for every conditional neighborhood U of x there exists α_0 such that $(x_\alpha)_{\alpha\geqslant\alpha_0} \sqsubseteq U$;

(ii) cluster point of (x_α) if for every conditional neighborhood U of x and every α there exists $\beta \geqslant \alpha$ such that $x_\beta \in U$.

We define $\mathrm{Lim}(x_\beta) = \sqcap\{\mathrm{cl}((x_\beta)_{\beta\geqslant\alpha}) : \alpha\}$.

Note that if x is a conditional limit or cluster point of a conditional net (x_α), then ax is a conditional limit or cluster point of the conditional net (ax_α), respectively. We indicate by $x_\alpha \to x$ that x is the conditional limit point of (x_α).

Proposition 4.24. *A $Y \in \mathcal{S}(X)$ is conditionally closed if and only if $x_\alpha \to x \in Y$ for every conditionally converging net $(x_\alpha) \subseteq Y$.*

Proposition 4.25. *Let X and Y be two conditional topological spaces and $f : X \to Y$ be a conditional function. Then the following assertions are equivalent:*

(i) f is conditionally continuous at x;

(ii) $f(x_\alpha) \to f(x)$ for every conditional net $x_\alpha \to x$;

Moreover, the composition of conditionally continuous functions is conditionally continuous.

We introduce the concept of conditional compactness.

Definition 4.26. Let X be a conditional topological space. A conditional open covering of X is a conditional family $(O^j) \subseteq \mathcal{T}$ such that $X \sqsubseteq \sqcup O^j$. We call X conditionally compact if for every conditional open covering (O^j) it holds

$$X \sqsubseteq \bigsqcup_{1\leqslant k\leqslant n} O^{j_k} \tag{4.2}$$

for some conditionally finite subfamily $(O^{j_k})_{1\leqslant k\leqslant n}$. Moreover, $Y \sqsubseteq X$ is conditionally compact if Y is conditionally compact with respect to the conditional relative topology on Y.

Proposition 4.27. *Let X be a conditional topological space. Then the following statements are equivalent:*

(i) X is conditionally compact;

(ii) every conditional net (x_α) has a conditional subnet (x_β) conditionally converging to some $x \in X$.

The following theorem is a conditional version of Tychonoff's theorem. The conditional product topology is defined as the conditional initial topology for the family (π_i), with $\pi^i : \prod X^i \to X^i$ by $\pi^i := (\pi^i_a)_{a\in\mathcal{A}}$ where $\pi^i_a : \prod X^i_a \to X^i_a$ is a projection for each $a \in \mathcal{A}$.

Theorem 4.28. *Let $(X^i, \mathcal{T}^i)$ be a family of conditional topological spaces and $X = \prod X^i$ be endowed with the conditional product topology. Then X is conditionally compact if and only if X^i is conditionally compact for every i.*

4.1.3 Conditional Fields and Metric Spaces

Definition 4.29. The triple $(X, +, \cdot)$ is a conditional ring if X is a conditional set, $+ : X \times X \to X$, $\cdot : X \times X \to X$ are conditional functions and $(X_1, +_1, \cdot_1)$ is a ring.

We denote $x \cdot y$ by xy. If there is no risk of confusion, we use the same notation for conditional addition and amalgamations, and for conditional multiplication and conditioning action, respectively. Further, we use the same notation 0 and 1 for the neutral elements of the conditional addition and multiplication and the distinguished elements of $\mathcal{A}$. As immediate consequence of the definition, it holds $a(x+y) = ax + ay$ and $a(x(y+z)) = axay + axaz = axy + axz$ for every $x, y, z \in X$ and $a \in \mathcal{A}$.

Let $(X, +, \cdot)$ be a conditional ring and I be a conditional set. Recall that $\mathcal{M}(I, X)$ denotes the conditional set of all conditional functions from I to X. Then $(\mathcal{M}(I, X), +, \cdot)$, where $(x+y)(i) := x(i) + y(i)$ and $(x \cdot y)(i) = x(i) \cdot y(i)$, is a conditional ring. Let $(S, +, \cdot)$ be a ring.

Definition 4.30. A conditional ring $(X, +, .)$ is a conditional field if for every $x \in \{0\}^{\sqsubset}$ there exists $y \in \{0\}^{\sqsubset}$ such that $xy = yx = 1$.

It holds that $\mathbf{Q}$ is a conditional field. Following the classical construction by Cauchy sequences, one can construct the conditional real numbers from $\mathbf{Q}$. We call the conditional ordered field $(\mathbf{R}, +, ., \leqslant)$ the conditional real numbers. We denote by $\mathbf{R}_+ := \{x \in \mathbf{R} : x \geqslant 0\}$ and $\mathbf{R}_{++} := \{x \in \mathbf{R} : x > 0\}$. In $\mathbf{R}$ every conditionally bounded subset has a conditional supremum and infimum. Furthermore the conditional topology generated by the conditional topological base

$$\mathcal{B} := \{B_\varepsilon(x) : x \in \mathbf{R}, \varepsilon \in \mathbf{R}_{++}\}, \quad B_\varepsilon(x) = \{y \in \mathbf{R} : |x - y| < \varepsilon\},$$

makes $\mathbf{R}$ to be a Hausdorff and complete[1] conditional topological space.

Definition 4.31. Let X be a conditional set. A conditional metric is a conditional function $d : X \times X \to \mathbf{R}_+$ such that

(i) $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for every $x, y \in X$;

(iii) $d(x, z) \leqslant d(x, y) + d(y, z)$ for every $x, y, z \in X$.

The pair (X, d) consisting of a conditional set X and metric d is called a conditional metric space. A conditional sequence $(x_n) \subseteq X$ is conditionally Cauchy if for every $\varepsilon \in \mathbf{R}_{++}$ there exists $n_0 \in \mathbf{N}$ such that $d(x_n, x_m) \leqslant \varepsilon$ for every $n, m \geqslant n_0$. Further, (X, d) is called conditionally complete if every conditional Cauchy sequence conditionally converges and conditionally sequentially compact if every conditional sequence $(x_n) \subseteq X$

[1] In the topological sense, that is, every conditional Cauchy sequence conditionally converges in $\mathbf{R}$.

has a conditional cluster point in X. A conditional set $Y \in \mathcal{S}(X)$ is conditionally bounded if there exists $M \in \mathbf{N}$ such that $d(x,y) \leqslant M$ for all $x, y \in Y$

The conditional version of the Heine-Borel Theorem is given as follows.

Theorem 4.32. *A conditional set $Y \in \mathcal{S}(\mathbf{R}^n)$ is conditionally closed and bounded if and only if it is conditionally compact or equivalently conditionally sequentially compact.*

4.2 Vector Spaces

Throughout this section we fix a nondegenerate $\mathcal{A}$ and consider only conditional sets on it.

Definition 4.33. Let X and K be two conditional sets. We call X a conditional K-module if K is a conditional ring, X_1 is a K_1-module and the scalar operation $\cdot : K \times X \to X$ is a conditional function. If K is a conditional field, X is a conditional K-vector space.

Let X be a conditional K-vector space, $Y, Z \sqsubseteq X$ and $\lambda \in K$. Then $Y + Z$ is defined as the image of $Y \times Z$ under the conditional addition in X, and λY as the image of $\{\lambda\} \times Y$ under the conditional scalar multiplication.

Recall that if $Y \in S(aX)$ and $Z \in S(bX)$, then $Y \times Z$ is defined as the conditional product of $(a \wedge b)Y$ and $(a \wedge b)Z$, and therefore $Y + Z \in S((a \wedge b)X)$. The same holds for λY. From now on we fix a conditional field K and consider conditional K-vector spaces.

Definition 4.34. Let X be a conditional vector space. Then $Y \in S(X)$ is a conditional subspace of X if

(i) $Y + Y \sqsubseteq Y$,

(ii) $\lambda Y \sqsubseteq Y$ for every $\lambda \in K$.

Moreover, we say that $Y \in S(X)$ conditionally spans X if $\mathrm{Span}(Y) = X$ where

$$\mathrm{Span}(Y) = \sqcap\{Z : Y \sqsubseteq Z, Z \text{ is a conditional subspace of } X\}.$$

Any conditional subspace of X has to contain 0 due to property (ii). Hence, for conditional subspaces (Y_i) it holds $\sqcap Y_i = \cap Y_i$. Since the conditional addition and the conditional scalar multiplication are conditional functions, it follows that $\lambda(\sqcap Y_i) = \sqcap(\lambda Y_i)$ and $\sqcap Y_i + \sqcap Y_i \sqsubseteq \sqcap(Y_i + Y_i)$ for any family (Y_i) and so $\mathrm{Span}(Y)$ is a conditional subspace of X. By definition, Y is a conditional subspace of X if and only if $\mathrm{Span}(Y) = Y$.

Definition 4.35. Given conditionally finite families $(x_k)_{1\leqslant k\leqslant n} \subseteq X$ and $(\lambda_k)_{1\leqslant k\leqslant n} \subseteq K$, where $n = \sum a_i n_i \in \mathbf{N}$, we define

$$\sum_{1\leqslant k\leqslant n} \lambda_k x_k := \sum a_i \left(\sum_{1\leq k_i \leq n_i} \lambda_{k_i} x_{k_i} \right).$$

Lemma 4.36. *Let X be a conditional vector space and $Y \in S(X)$. Then it holds*

$$\mathrm{Span}(Y) = \left\{ \sum_{1\leqslant k\leqslant n} \lambda_k x_k : (x_k)_{1\leqslant k\leqslant n} \subseteq Y, (\lambda_k)_{1\leqslant k\leqslant n} \subseteq K, n \in \mathbf{N} \right\}. \tag{4.3}$$

Proof. If Z is a conditional subspace of X such that $Y \sqsubseteq Z$ it holds that $\lambda x \in Z$ and $x + y \in Z$ for any $x, y \in Y$ and $\lambda \in K$. By induction, it follows that $\sum_{1\leq k\leq n} \lambda_k x_k \in Z$ for $(\lambda_k)_{1\leq k\leq n} \subseteq K$ and $(x_k)_{1\leq k\leq n} \subseteq Y$ where $n \in \mathbb{N}$. Since Z is a conditional set, it holds that $\sum_{i\in I} a_i \sum_{1\leq k_i\leq n_i} \lambda_{k_i} x_{k_i} \in Z$ for $(a_i)_{i\in I} \in \mathcal{K}(1)$, $(\lambda_{k_i})_{1\leq k_i\leq n_i} \subseteq K$ and $(x_{k_i})_{1\leq k_i\leq n_i} \subseteq Y$ where $(n_i)_{i\in I} \subseteq \mathbb{N}$. Since $\mathrm{Span}(Y)$ is the conditional intersection of all conditional subspaces conditionally including Y, it follows that the set on the right-hand side of (4.3), denoted by M, is conditionally included in $\mathrm{Span}(Y)$. For the reverse implication, we have to show that M is a conditional subspace of X. By definition M is in $S(X)$. Let $\sum a_i(\sum_{1\leq k_i\leq n_i} \lambda_{k_i} x_{k_i})$ and $\sum b_j(\sum_{1\leq l_j\leq m_j} \lambda_{l_j} y_{l_j})$ be in M. Since the conditional scalar multiplication is a conditional function, it follows that $\lambda \sum a_i(\sum_{1\leq k_i\leq n_i} \lambda_{k_i} x_{k_i}) = \sum a_i(\sum_{1\leq k_i\leq n_i} \lambda\lambda_{k_i} x_{k_i})$ which is in M for any $\lambda \in K$. Moreover,

$$\begin{aligned}\sum_{i\in I} a_i \left(\sum_{1\leq k_i\leq n_i} \lambda_{k_i} x_{k_i} \right) &+ \sum b_j \left(\sum_{1\leq l_j\leq m_j} \lambda_{l_j} y_{l_j} \right) \\ &= \sum_{i\in I, j\in J} a_i \wedge b_j \left(\sum_{1\leq k_i\leq n_i} \lambda_{k_i} x_{k_i} + \sum_{1\leq l_j\leq m_j} \lambda_{l_j} y_{l_j}) \right)\end{aligned}$$

is in M. □

4.2.1 Topological Vector Spaces

In the remaining of this chapter we examine conditional $\mathbf{R}$-vector spaces.

Definition 4.37. Let X be a conditional vector space. For $Y \in \mathcal{S}(X)$, we define

$$\mathrm{conv}(Y) := \{\lambda x + (1-\lambda) y : x, y \in Y, \lambda \in \mathbf{R} \text{ and } 0 \leqslant \lambda \leqslant 1\}.$$

Then a conditional set $Y \in \mathcal{S}(X)$ is conditionally

- convex if $Y = \operatorname{conv}(Y)$;
- circled if $\lambda y \in Y$ for every $y \in Y$ and $\lambda \in \mathbf{R}$ such that $|\lambda| \leqslant 1$;
- absorbing if for any $x \in X$ there exists $\lambda_x \in \mathbf{R}_{++}$ such that $\lambda x \in Y$ for any $\lambda \in \mathbf{R}$ with $|\lambda| \leqslant \lambda_x$.

A conditional vector space X endowed with a conditional topology $\mathcal{T}$ is a conditional topological vector space if the conditional functions $+ : X \times X \to X$ and $\cdot : \mathbf{R} \times X \to X$ are conditionally continuous. If further X has a conditional neighborhood base of 0 consisting only of conditionally convex sets, then X is a conditional locally convex topological vector space. In a conditional topological vector space $(X, \mathcal{T})$ we call $Y \in S(X)$ conditionally $\mathcal{T}$-bounded if for every conditional neighborhood U of 0 there exists $\lambda \in \mathbf{R}$ such that $Y \sqsubseteq \lambda U$.

Definition 4.38. Let X be a conditional vector space. Then a conditional function $f : X \to \mathbf{R}$ is conditionally linear if it holds that

$$f(\lambda x + \mu y) = \lambda f(x) + \mu f(y),$$

for every $x, y \in X$ and any $\lambda, \mu \in \mathbf{R}$.

Lemma 4.39. *In a conditional topological vector space $(X, \mathcal{T})$ it holds that*

(i) λO is conditionally open if O is conditionally open and $\lambda \in \{0\}^{\sqsubset}$;

(ii) λC is conditionally closed if C is conditionally closed and $\lambda \in \mathbf{R}$;

(iii) λC is conditionally compact if C is conditionally compact and $\lambda \in \mathbf{R}$;

(iv) $Y + O = \operatorname{cl}(Y) + O$ for every conditionally open $O \in S(X)$ and any $Y \in S(X)$;

(v) $Y + C$ is conditionally closed if $Y \in S(X)$ is conditionally compact and $C \in S(X)$ is conditionally closed;

(vi) $Y + C$ is conditionally compact if $Y \in S(X)$ and $C \in S(X)$ are conditionally compact;

(vii) a conditionally linear function $f : X \to \mathbf{R}$ is conditionally continuous if it is conditionally continuous at 0.

Proof. Without loss of generality, we may assume that all sets we consider are in $\mathcal{S}(X)$. The proof of *(i)*, *(ii)* and *(iii)* follows by conditional continuity of the conditional scalar multiplication.

For the proof of *(iv)*, it holds that $Y \sqsubseteq \operatorname{cl}(Y)$ and hence $Y + O \sqsubseteq \operatorname{cl}(Y) + O$. Conversely, let $x = y + z \in \operatorname{cl}(Y) + O$. Since O is conditionally open, there exists a conditional neighborhood U of 0 such that $z + U \sqsubseteq O$, due to Proposition 4.18. Moreover, there exists $\tilde{y} \in X$ such that $\tilde{y} \in Y \sqcap (y + U)$, since $y \in \mathrm{cl}(Y)$. Hence, $x = y + z = \tilde{y} + z + (y - \tilde{y}) \in \tilde{y} + z + U \sqsubseteq Y + O$.

To prove *(v)* let $(x_\alpha) = (y_\alpha + z_\alpha)$ be a conditional net in $Y + C$ conditionally converging to x. Since Y is conditionally compact, there exists a conditional subnet (y_β) of (y_α) such that $y_\beta \to y \in Y$, due to Proposition 4.27. The conditional continuity of the conditional addition yields $z_\beta = x_\beta - y_\beta \to x - y =: z$ and since C is conditionally closed Proposition 4.24 implies $z \in C$. Hence, $x = y + z \in Y + C$ and by Proposition 4.24 it follows that $Y + C$ is conditionally closed. Property *(vi)* can be proven by using the one-to-one relation of conditional compactness and conditional nets, as in Proposition 4.27.

The proof of *(vii)* follows by the characterization in Proposition 4.25. Let f be conditionally continuous at 0. Then consider a conditional net $x_\alpha \to x$. This implies $x_\alpha - x \to 0$ and by conditional continuity of f at 0 it holds $f(x_\alpha - x) \to f(0)$. By conditional linearity of f it holds that $f(0) = 0$ and $f(x_\alpha - x) = f(x_\alpha) - f(x)$ for every α. Hence, $f(x_\alpha) \to f(x)$ which shows that f is also conditionally continuous at x. $\square$

Proposition 4.40. *For a conditional topological vector space X there exists a conditional neighborhood base $\mathcal{B}_0$ of 0 consisting only of conditionally closed, absorbing and circled conditional sets such that for every $U^1 \in \mathcal{B}_0$ there exists $U^2 \in \mathcal{B}_0$ with $U^2 + U^2 \sqsubseteq U^1$.*

Conversely, let X be a conditional vector space. If there exists a conditional filter base $\mathcal{B}_0$ of conditionally absorbing, circled conditional subsets of X containing 0 such that for every $U^1 \in \mathcal{B}_0$ there exists $U^2 \in \mathcal{B}_2$ with $U^2 + U^2 \sqsubseteq U^1$, then $\mathcal{T}^{\mathcal{B}}$ where $\mathcal{B} := \{x + \mathcal{B}_0 : x \in X\}$, makes X to be a conditional topological vector space.

Proof. *Step 1:* Suppose that X is a ctvs. Let U be a conditional neighborhood of 0. By means of *(i)*, λU is a conditional neighborhood of 0 for every $\lambda \in \mathbf{R}^*$. The conditional continuity of the conditional scalar multiplication implies that there exists $\lambda_0 \in \mathbf{R}_{++}$ and a conditional neighborhood $\tilde{U}$ of 0 such that $\lambda \tilde{U} \sqsubseteq U$ for every $\lambda \in \mathbf{R}$ with $|\lambda| \leqslant \lambda_0$. The conditional set $V = \sqcup_{|\lambda| \leqslant \lambda_0} \lambda \tilde{U}$ is conditionally absorbing by definition, conditionally circled and it holds $V \sqsubseteq U$. Furthermore, the conditional continuity of the conditional addition at $(0, 0)$ implies that there exists a conditionally open neighborhood W of 0 such that $W + W \sqsubseteq U$. Hence, $\mathrm{cl}(W) + \mathrm{cl}(W) \sqsubseteq \mathrm{cl}(U)$. Indeed, by Proposition 4.24 we have that $x \in \mathrm{cl}(W)$ if there exists a net $(x_\alpha) \in W$ such that $x_\alpha \to x$. Hence, for $x + y$ in $\mathrm{cl}(W) + \mathrm{cl}(W)$ there exist conditional nets (x_α) and (y_α) in W such that $x_\alpha \to x$ and $y_\alpha \to y$. Therefore $x_\alpha + y_\alpha \in U$ and since $x_\alpha + y_\alpha \to x + y$, it follows $x + y \in \mathrm{cl}(U)$. Moreover, note that the conditional closure of a conditionally circled and absorbing set is again conditionally circled and absorbing. Indeed, by Proposition 4.24 for $y \in \mathrm{cl}(Y)$ there exists a conditional net (y_α) in Y such that $y_\alpha \to y$. Therefore, $\lambda y_\alpha \to \lambda y$ and $\lambda y \in Y$ for any $|\lambda| \leqslant 1$ since Y is conditionally circled. Therefore,

$\lambda y \in \mathrm{cl}(Y)$. The property that $\mathrm{cl}(Y)$ is conditionally absorbing can be shown by the same argument.

Concluding, starting with $\mathcal{U}(0)$ which is a collection of conditional neighborhoods of 0 we can define $\mathcal{B}_0 = \{\mathrm{cl}\left(\sqcup_{|\lambda|\leqslant\lambda_0}\lambda U\right) : U \in \mathcal{U}(0), \lambda_0 \in \mathbf{R}_{++}\}$, which consists only of conditionally closed, absorbing and circled neighborhoods of 0 and for $U_1 \in \mathcal{B}_0$ there exists $U_2 \in \mathcal{B}_0$ such that $U_2 + U_2 \sqsubseteq U_1$. To prove that $\mathcal{B}_0$ is a conditional neighborhood base we first notice that by the previous examination we ensure that for any conditional neighborhood U of 0 there is $\tilde{U} \in \mathcal{B}_0$ with $\tilde{U} \sqsubseteq U$. That $\mathcal{B}_0 \in S(S(X))$ is due to the conditional continuity of the conditional scalar multiplication, and to the property that the conditional closure is a conditional operation.

Step 2: Let $\mathcal{B}_0$ be a conditional filter base of conditional subsets containing 0 of conditionally absorbing and circled sets. Since the conditional addition is a conditional function it holds that

$$\mathcal{B} := \{x + U : U \in \mathcal{B}_0, x \in X\}$$

is in $S(S(X))$ and defines a conditional topological base on X. We have to show that $(X, \mathcal{T})$ is a ctvs, where $\mathcal{T}^{\mathcal{B}} = \mathcal{T}$.

To show that the conditional addition is conditionally continuous, fix $x_0, y_0 \in X$ and $U \in \mathcal{B}_0$. Then there exists $V \in \mathcal{B}_0$ such that $V + V \sqsubseteq U$. Therefore, if $x \in x_0 + V$ and $y \in y_0 + V$ it follows that $x + y \in x_0 + y_0 + U$ and hence the conditional addition is conditionally continuous in (x_0, y_0). To show the conditional continuity of the conditional scalar multiplication, fix $\mu_0 \in \mathbf{R}$, $x_0 \in X$ and $U \in \mathcal{B}_0$. Then there exists $V \in \mathcal{B}_0$ such that $V + V \sqsubseteq U$. Since V is conditionally absorbing it holds that there exists an $\lambda_0 \in \mathbf{R}_{++}$ such that $\lambda x_0 \in V$ for all $|\lambda| \leq \lambda_0$. It is sufficient to argue by conditional sequences. Pick $n \in \mathbf{N}$ such that $|\mu_0| + \lambda_0 < n$. For $\mu \in \mathbf{R}$ with $|\mu - \mu_0| < \lambda_0$ it holds that $1/(n\,|\mu|) \leqslant 1/(n\,|\mu_0| + n\lambda_0) < 1$. Hence, for $x \in x_0 + 1/nV$ it follows that

$$\mu x = \mu_0 x_0 + (\mu - \mu_0)x_0 + \mu(x - x_0) \in \mu_0 x_0 + V + \mu\frac{1}{n}V \sqsubseteq \mu_0 x_0 + V + V \sqsubseteq \mu_0 x_0 + U,$$

where we used the property that V is conditionally circled. Hence, the conditional scalar multiplication is conditionally continuous at (μ_0, x_0).

□

Corollary 4.41. *In a conditional topological vector space X, the collection of all conditionally open, circled neighborhoods of* 0 *is a conditional neighborhood base of* 0*.*

Proof. It is sufficient to show that the conditional interior of a conditionally circled neighborhood of 0 is again conditionally circled. Let V be a conditionally circled neighborhood of 0. Note that 0 is in the conditional interior of V. Pick $y \in \mathrm{int}(V)$ and $|\lambda| \leqslant 1$. Let W be a conditional neighborhood of 0 such that $y + W \sqsubseteq V$. Then

$\lambda y+\lambda W=\lambda(y+W)\sqsubseteq \lambda V\sqsubseteq V$, since V is conditionally circled. Therefore, $\lambda y\in \operatorname{int}(V)$ which shows that also the conditional interior of V is conditionally circled. □

Definition 4.42. Let X be a conditional vector space. Then a conditional function $f : X \to \mathbf{R}$ is conditionally convex if it holds that

$$f(\lambda x + (1-\lambda)y) \leqslant \lambda f(x) + (1-\lambda)f(y),$$

for every $x, y \in X$ and any $\lambda \in \mathbf{R}$ with $0 \leqslant \lambda \leqslant 1$.

Next we prove the conditional version of the Hahn-Banach Theorem.

Theorem 4.43. *Let X be a conditional vector space and $p : X \to \mathbf{R}$ be a conditionally convex function. Moreover, let $Y \in \mathcal{S}(X)$ be a conditional subspace of X and $f : Y \to \mathbf{R}$ be a conditionally linear function such that $f(y) \leqslant p(y)$ for every $y \in Y$. Then there exists a conditionally linear function $\hat{f} : X \to \mathbf{R}$ such that $\hat{f}(x) \leqslant p(x)$ for every $x \in X$ and $\hat{f}(y) = f(y)$ for all $y \in Y$.*

Proof. Define $\mathcal{E}$ as the collection of all pairs (h, H) where H is a conditional vector space with $Y \sqsubseteq H \sqsubseteq X$ and $h : H \to \mathbf{R}$ is a conditionally linear function such that $h(y) = f(y)$ for every $y \in Y$ and $h(x) \leqslant p(x)$ for every $x \in H$. It follows that $\mathcal{E}$ is a conditional family. Furthermore, the conditional relation $(h, H) \leqslant (h', H')$ defined by $H \sqsubseteq H'$ and $h'(x) = h(x)$ for all $x \in H$ is a conditional partial order. Given a chain (h_α, H_α) in $\mathcal{E}$, it follows that $H := \cup H_\alpha = \sqcup H_\alpha$ is a conditional subspace of X. Setting $h(x) = \sum a_i h_{\alpha_i}(x_i)$ for $x = \sum a_i x_i \in H$, where $(a_i) \in \mathcal{K}(1)$ and $x_i \in H_{\alpha_i}$ for all i, defines a conditional linear function $h : H \to X$, since $\mathcal{E}$ is a conditional family. Hence, $(h, H) \in \mathcal{E}$, so that the chain has a maximal element. By Zorn's Lemma, there exists a maximal element $(\hat{f}, \hat{H})$ in $\mathcal{E}$. Let us show that $\hat{H} = X$.

By contradiction, suppose that $\hat{H}^{\sqsubset} \in \mathcal{S}(aX)$ for some $a > 0$. Without loss of generality, we assume that $a = 1$. Let $v \in \hat{H}^{\sqsubset}$ and define $\tilde{H} := \{x + \lambda v : x \in \hat{H}, \lambda \in \mathbf{R}\}$ which is a conditional subspace of X such that $Y \sqsubseteq \hat{H} \sqsubset \tilde{H} \sqsubseteq X$. Every $y \in \tilde{H}$ is of the form $y = x + \lambda v$ for a unique $x \in \hat{H}$ and $\lambda \in \mathbf{R}$. Indeed, suppose that $y = x + \lambda v = \tilde{x} + \tilde{\lambda} v$, it follows that $x - \tilde{x} = (\tilde{\lambda} - \lambda)v$. However, $x - \tilde{x} \in \hat{H}$ from which follows that $(\tilde{\lambda} - \lambda)v \in \hat{H}$. Let $b = \vee\{a \in \mathcal{A} : a\lambda \neq a\tilde{\lambda}\}$. Then it follows that $bv \in b\hat{H}$ contradicting the fact that $v \in \hat{H}^{\sqsubset}$. Hence, $b = 0$ and therefore $\lambda = \tilde{\lambda}$ and $x = \tilde{x}$.

Any linear extension $\tilde{f}$ of $\hat{f}$ to $\tilde{H}$ has to fulfill $\tilde{f}(x + \lambda v) = \hat{f}(x) + \lambda \tilde{f}(v)$ for every $x \in \hat{H}$ and $\lambda \in \mathbf{R}$. So it is sufficient to prove the existence of $r \in \mathbf{R}$ such that $\hat{f}(x) + \lambda r \leqslant p(x + \lambda v)$ for all $x \in \hat{H}, \lambda \in \mathbf{R}$. Pick $\lambda \in \mathbf{R}$. Then we can find a partition a_1, a_2, a_3 such that $a_1\lambda > a_1 0$, $a_2\lambda < a_2 0$ and $a_3\lambda = a_3 0$. Then it holds that

$\hat{f}(x) + \lambda r \leqslant p(x + \lambda v)$ reads as

$$a_1 r \leqslant a_1 \left(\frac{1}{a_1\lambda}[\, p(x + a_1\lambda v) - \hat{f}(x)] \right), \tag{4.4}$$

$$a_2 r \leqslant a_2 \left(-\frac{1}{a_2\lambda}[\hat{f}(x) - p(x + a_2\lambda v)] \right), \tag{4.5}$$

$$a_3 \hat{f}(x) \leqslant a_3 p(x), \tag{4.6}$$

for every $x \in \hat{H}$. Inequality (4.6) is fulfilled by definition of $\hat{f}$. The inequalities (4.4) and (4.5) hold for any $\lambda \in \mathbf{R}$ if and only if

$$\frac{1}{\kappa}[\hat{f}(x) - p(x - \kappa v)] \leqslant r \leqslant \frac{1}{\mu}[\, p(y + \mu v) - \hat{f}(y)], \tag{4.7}$$

for all $x, y \in \hat{H}$ and $\kappa, \mu \in \mathbf{R}$ with $\kappa > 0, \mu > 0$. This is equivalent to

$$\frac{1}{\kappa}[\hat{f}(x) - p(x - \kappa v)] \leqslant \frac{1}{\mu}[\, p(y + \mu v) - \hat{f}(y)],$$

for all $x, y \in \hat{H}$ and $\kappa, \mu \in \mathbf{R}$ with $\kappa > 0, \mu > 0$. Reorganizing the previous inequality, gives $\hat{f}(\mu x + \kappa y) \leqslant \mu p(x - \kappa v) + \kappa p(y + \mu v)$ which is always fulfilled. Indeed,

$$\hat{f}(\mu x + \kappa y) = (\kappa + \mu)\hat{f}\left(\frac{\mu}{\kappa + \mu}x + \frac{\kappa}{\kappa + \mu}y \right) \leqslant (\kappa + \mu)p\left(\frac{\mu}{\kappa + \mu}x + \frac{\kappa}{\kappa + \mu}y \right),$$

since $\mu/(\kappa + \mu)x + \kappa/(\kappa + \mu)y \in \hat{H}$ as $x, y \in \hat{H}$. Moreover, by conditional convexity of p it follows

$$\begin{aligned}(\kappa + \mu)p\left(\frac{\mu}{\kappa + \mu}x + \frac{\kappa}{\kappa + \mu}y \right) &= (\kappa + \mu)p\left(\frac{\mu}{\kappa + \mu}[x - \kappa v] + \frac{\kappa}{\kappa + \mu}[y + \mu v] \right) \\ \leqslant (\kappa + \mu)\left(\frac{\mu}{\kappa + \mu}p(x - \kappa v) + \frac{\kappa}{\kappa + \mu}p(y + \mu v) \right) &= \mu p(x - \kappa v) + \kappa p(y + \mu v).\end{aligned}$$

Thus, (4.7) has to be fulfilled for some $r \in \mathbf{R}$. This means, that if there exists $v \in \hat{H}^{\sqsubset}$ we can extend $\hat{f}$ which is a contradiction. □

Theorem 4.44. *Let X be a conditional locally convex topological vector space and $C^1, C^2 \in \mathcal{S}(X)$ be two conditionally convex sets such that $C^1 \sqcap C^2 = \mathbf{0}$.*

(i) If C^1 is conditionally open, then there exists a conditionally continuous linear function $f : X \to \mathbf{R}$ such that

$$f(x) < f(y), \quad \text{for every } x \in C^1 \text{ and } y \in C^2.$$

(ii) If C^1 conditionally compact and C^2 conditionally closed, then there exists a con-

ditionally continuous linear function $f : X \to \mathbf{R}$ and $\varepsilon \in \mathbf{R}_{++}$ such that

$$f(x) + \varepsilon < f(y), \quad \text{for every } x \in C^1 \text{ and } y \in C^2.$$

Proof. Separation theorems for L^0-modules have been proven in [43]. With exactly the same methods it can be extended to any conditional $\mathbf{R}$-vector space. □

4.2.2 Duality

Definition 4.45. Two conditional vector spaces X and X' are said to be a conditional dual pair, denoted by $\langle X, X' \rangle$ if there exists a conditional functional $\langle \cdot, \cdot \rangle : X \times X' \to \mathbf{R}$ such that the following two properties are fulfilled:

(i) $x \mapsto \langle x, x' \rangle$ for every fixed $x' \in X'$, and $x' \mapsto \langle x, x' \rangle$ for every fixed $x \in X$, are conditionally linear;

(ii) $\langle \cdot, x' \rangle \equiv 0$ and $\langle x, \cdot \rangle \equiv 0$ implies $x' = 0$ and $x = 0$, respectively.

For a conditional dual pair $\langle X, X' \rangle$, we denote by $\sigma(X, X')$ the conditional initial topology for the conditional functions $\langle \cdot, x' \rangle : X \to \mathbf{R}$, $x \mapsto \langle x, x' \rangle$ for every $x' \in X'$. Moreover, we denote by $\sigma(X', X)$ the conditional initial topology for the conditional functions $\langle x, \cdot \rangle : X' \to \mathbf{R}$, $x' \mapsto \langle x, x' \rangle$ for every $x \in X$.

Note that both $(\langle x, \cdot \rangle)_{x \in X}$ and $(\langle \cdot, x' \rangle)_{x' \in X'}$ are conditional families. We define $\mathcal{M}(X, \mathbf{R})$ as the conditional set of all conditional functions from X to $\mathbf{R}$. It holds that X' is a conditional subspace of $\mathcal{M}(X, \mathbf{R})$ and the conditional product topology on $\mathcal{M}(X, \mathbf{R})$ induces the conditional $\sigma(X', X)$-topology as a conditional relative topology. Similar to the standard case, $\mathcal{M}(X, \mathbf{R})$ is a conditionally Hausdorff locally convex topological vector space. Hence, it follows that also $(X', \sigma(X', X))$ is a conditionally Hausdorff locally convex topological vector space and analogously also $(X, \sigma(X, X'))$.

Definition 4.46. For a conditional topological vector space $(X, \mathcal{T})$ we define $X^\star = (X, \mathcal{T})^\star$ to be the conditional vector space of all conditionally continuous linear functions $x^\star : X \to \mathbf{R}$. We say $\mathcal{T}$ is compatible with the conditional dual pair $\langle X, X' \rangle$ if $(X, \mathcal{T})^\star = X'$.

For a conditionally linear function $f : X \to \mathbf{R}$ on a conditional vector space X we define the conditional kernel $\ker(f)$ as the conditional preimage of 0. Since $0 \in \ker(f)$, it holds that $\ker(f) = \{x \in X : f(x) = 0\}$ is in $S(X)$. Next we prove the conditional version fo the fundamental theorem of duality.

Theorem 4.47. *Let f be a conditionally linear function on a conditional vector space X and $(f_k)_{1 \leqslant k \leqslant n}$ a conditionally finite family of conditionally linear functions on X. Then the following is equivalent:*

(i) There exists a conditional family $(\lambda_k)_{1\leqslant k\leqslant n} \subseteq \{0\}^{\sqsubset}$ *such that* $f = \sum_{1\leqslant k\leqslant n} \lambda_k f_k$.

(ii) $\bigsqcap_{1\leqslant k\leqslant n} \ker(f_k) \sqsubseteq \ker(f)$.

Proof. Recall that if $n = \sum a_i n_i$, for $[a_i, n_i]_{i\in\mathbb{N}} \subseteq \mathcal{A} \times \mathbb{N}$, then any k contributing to the conditional sum indexed by $1 \leqslant k \leqslant n$ is of the form $k = \sum a_i k_i$ with $1 \leq k_i \leq n_i$. Moreover, $f_k = \sum a_i f_{k_i}$. Thus, $f = \sum_{1\leqslant k\leqslant n} \lambda_k f_k$ means $f = \sum_{i\in\mathbb{N}} a_i(\sum_{k_i=1}^{n_i} \lambda_{k_i} f_{k_i})$, with $\lambda_{k_i} \in \{0\}^{\sqsubset}$.

Let $f = \sum_{1\leqslant k\leqslant n} \lambda_k f_k$. For $x \in \bigsqcap_{1\leqslant k\leqslant n} \ker(f_k)$ we know $f_k(x) = 0$ for all $1 \leqslant k \leqslant n$ and hence $f(x) = 0$, which means $x \in \ker(f)$.

To show the reverse implication suppose $\bigsqcap_{1\leqslant k\leqslant n} \ker(f_k) \sqsubseteq \ker(f)$. Define $T : X \to \mathbf{R}^n$ as $T(x) = (f_k(x))_{1\leqslant k\leqslant n}$. This is a conditionally linear operator, since any f_k is so. If $f_k(x) = f_k(y)$ for every $1 \leqslant k \leqslant n$, it follows that $f_k(x - y) = 0$ for every $1 \leqslant k \leqslant n$. Since $\bigsqcap_{1\leqslant k\leqslant n} ker(f_k) \sqsubseteq ker(f)$, it follows that $f(x - y) = 0$ and therefore $f(x) = f(y)$. Thus, the conditional function $\phi : T(X) \to \mathbf{R}$, defined by $\phi((f_k(x))_{1\leqslant k\leqslant n}) = f(x)$ is well-defined and a conditionally linear function.

We can conditionally extend ϕ to the entire $\mathbf{R}^n$. Indeed, let e_k, for $1 \leqslant k \leqslant n$, be the conditional vector with entry 1 at position k and entry 0 at any other position. By conditional linearity of ϕ, it follows that $\phi((f_k(x))_{1\leqslant k\leqslant n}) = \sum_{1\leqslant k\leqslant n} f_k(x)\phi(e_k)$ for any $x \in X$. Moreover, if every $\phi(e_k)$ exists, then by linearity of ϕ it follows that ϕ is a conditional function on the whole $\mathbf{R}^n$. If not, define $J = \{k : 1 \leqslant k \leqslant n, \phi(e_k)$ does not exist on some $a_k > 0\}$ to be the set of conditional indexes for which the image of e_k does not exist on 1 and $N(b) := \{k : 1 \leqslant k \leqslant n, a_k \geq b\}$ for $b \in \mathcal{A}$. Define $a = \vee_{k\in J} a_k$. By Assumption (P) we can find $(b_j) \in \mathcal{K}(a)$ with $b_j \leq a_k$ for some k. For every b_j fix one m_j with $m_j \in \mathcal{N}(b_j)$. If $\mathcal{N}(b_j)$ is a singleton then just define $b_j\phi(e_{m_j}) = b_j 1$. Otherwise, define $b_j\phi(e_k) = b_j 1$ for all $k \in N(b_j)$ with $b_j k \neq b_j m_j$ and $b_j\phi(e_{m_j}) = b_j(f(x) - \sum_{1\leqslant k\leqslant n, b_j k\neq b_j m_j} f_k(x)\phi(e_k))$ for some $x \in (\sqcap_{1\leqslant k\leqslant n} \ker f_k)^{\sqsubset}$. Define $d = \vee\{c :$ there exists $x \in c(\sqcap_{1\leqslant k\leqslant n} \ker f_k)^{\sqsubset}\}$. On d we can apply the procedure explained above. On d^c we know that all $d^c f_k$ and $d^c f$ are the conditional functions which is constant equal to $d^c 0$ and the claim is trivially fulfilled there.

In this way we can find $(\lambda_k)_{1\leqslant k\leqslant n} \subseteq \{0\}^{\sqsubset}$, such that $f((y_k)_{1\leqslant k\leqslant n}) = \sum_{1\leqslant k\leqslant n} \lambda_k y_k$ for every $(y_k)_{1\leqslant k\leqslant n} \in \mathbf{R}^n$, by setting $\lambda_k = \phi(e_k)$. Therefore, $f(x) = \sum_{1\leqslant k\leqslant n} \lambda_k f_k(x)$. □

Corollary 4.48. *If* $\langle X, X'\rangle$ *is a dual pair, then it holds that* $(X, \sigma(X, X'))^\star = X'$ *and* $(X', \sigma(X', X))^\star = X$.

Proof. Let $f : X \to \mathbf{R}$ be a conditionally linear, $\sigma(X, X')$-closed function. Since f is conditionally $\sigma(X, X')$-continuous at 0, there exists a conditional neighborhood U of 0 in $\sigma(X, X')$ such that $f(U) \sqsubseteq B_1(0)$. Therefore, there exists a conditional family $(x'_k)_{1\leqslant k\leqslant n}$ in X' and $\varepsilon \in \mathbf{R}_{++}$ such that if for all $1 \leqslant k \leqslant n$ it holds $|\langle x, x'_k\rangle| \leqslant \varepsilon$, then it follows that $x \in U$ and hence $|f(x)| \leqslant 1$. Define $f_k = \langle ., x'_k\rangle$ for every $1 \leqslant k \leqslant n$ and let

$x \in \bigsqcap_{1 \leqslant k \leqslant n} \ker(f_k)$. Therefore, $\langle x, x'_k \rangle = 0$, and hence for any $t \in \mathbf{R}$ also $\langle tx, x'_k \rangle = 0$, for every $1 \leqslant k \leqslant n$. Therefore, $|tf(x)| \leqslant 1$ for any t which implies $f(x) = 0$ and hence $x \in \ker(f)$. Thus, we can apply Theorem 4.47 and obtain that $f(x) = \sum_{1 \leqslant k \leqslant n} \lambda_k x'_i$ for some $(\lambda_k)_{1 \leqslant k \leqslant n} \subseteq \{0\}^{\sqsubseteq}$ which means $f \in X'$, since X' is a conditional vector space. $\square$

The previous theorem shows that $\sigma(X, X')$ and $\sigma(X', X)$ are compatible with the underlying conditional dual pair.

Lemma 4.49. *Let $\langle X, X' \rangle$ be a conditional dual pair. Then $Y \in S(X)$ is conditionally $\sigma(X, X')$-bounded if and only if for each $x' \in X'$ there exists $\lambda_{x'} \in \mathbf{R}_{++}$ such that $|\langle x, x' \rangle| \leqslant \lambda_{x'}$ for every $x \in Y$. Likewise, $Y' \in S(X')$ is conditionally $\sigma(X', X)$-bounded if and only if for each $x \in X$ there exists $\lambda_x \in \mathbf{R}_{++}$ such that $|\langle x, x' \rangle| \leqslant \lambda_x$ for every $x' \in Y'$.*

Proof. We will only prove the first claim, since the second follows by the same argumentation. Let Y be conditionally $\sigma(X, X')$-bounded. Fix $x' \in X'$. Consider $U := \{x \in X : |\langle x, x' \rangle| < 1\}$ which is a conditional neighborhood of 0, since $\langle ., x' \rangle$ is conditionally $\sigma(X, X')$-continuous and linear. Due to the conditional boundedness of Y, there exists $\lambda \in \mathbf{R}_{++}$ such that $Y \sqsubseteq \lambda U$. Therefore, $Y \sqsubseteq \{x \in X : |\langle x, x' \rangle| < \lambda\}$. Hence, $|\langle x, x' \rangle| < \lambda$ for any $x \in Y$.

For the other implication fix a conditional neighborhood U of 0. The claim follows if there exists $\lambda \in \mathbf{R}_{++}$ such that $\{x \in Y : |\langle x, x' \rangle| \leqslant \lambda\} \sqsubseteq U$ for some $x' \in X'$. Indeed, since there exists $\lambda_{x'} \in \mathbf{R}_{++}$ such that $Y \sqsubseteq \{x \in X : |\langle x, x' \rangle| \leqslant \lambda_{x'}\}$, it would follow that $Y \sqsubseteq (\lambda_{x'}/\lambda)U$. To prove the existence of such λ we use to the definition of the conditional topology $\sigma(X, X')$. As a conditional initial topology we know that every conditionally open conditional neighborhood is of the form $U = \sum a_n (\bigsqcap_{j_n \in J_n} (x'_{j_n})^{-1}(O_{j_n}))$ with $(a_n) \in \mathcal{K}(1)$, every J_n is conditionally finite and every $O_{j_n} \sqsubseteq \mathbf{R}$ is conditionally open. Now fix some a_n and j_n. Then by definition of the conditional Euclidean topology in $\mathbf{R}$, and since O_{j_n} is a conditional neighborhood of 0 there exists a conditional ball $B_{\mu_n}(0) \sqsubseteq O_{j_n}$ [2]. Since x'_{j_n} is a conditional function, it follows that $x'^{-1}_{j_n}(B_{\mu_n}(0)) \sqsubseteq x'^{-1}_{j_n}(O_{j_n})$ which is $\{x \in X : \left|\langle x, x'_{j_n} \rangle\right| \leqslant \mu_n\} \sqsubseteq x'^{-1}_{j_n}(O_{j_n})$. Thus, $\{a_n x \in a_n Y : \left|\langle x, x'_{j_n} \rangle\right| \leqslant \mu_{j_n}\} \sqsubseteq a_n (x'^{-1}_{j_n}(O_{j_n}) \sqcap Y)$. This can be done for every j_n. Define $\mu_n = \min\{\mu_{j_n} : j_n \in J_n\}$. Since we apply the conditional minimum of conditionally finitely many conditional real numbers strictly greater then $a_n 0$, it follows that $a_n \mu_n > a_n 0$. Moreover, there is a conditionally linear function x'_n attaining this $a_n \mu_n$. Indeed, as the conditional minimum of conditionally finitely many numbers, there exists $(b_m) \in \mathcal{K}(a_n)$ and $(x'_{j_n^m}) \subseteq (x'_{j_n})$ such that for $x'_n := \sum b_m x'_{j_n^m}$ it holds $\lambda_{x'_n} = \mu_n$. By construction, x'_n is a conditionally linear function on a_n. Moreover, $a_n 0 \in a_n (x'^{-1}_{j_n}(O_{j_n}) \sqcap Y)$ for every j_n. Altogether, it follows

[2] Use the definitions and the fact that if $0 \in B_\varepsilon(y)$, for $\varepsilon \in \mathbf{R}_{++}$ and $y \in \mathbf{R}$, then there exists $\varepsilon' \in \mathbf{R}_{++}$ with $B_{\varepsilon'}(0) \sqsubseteq B_\varepsilon(y)$, namely $\varepsilon' := \varepsilon - |y|$.

that $\{a_n x \in a_n Y : a_n |\langle x, x'_n\rangle| \leqslant \mu_n\} \sqsubseteq a_n U$. Doing so for every n we can define $x' := \sum a_n x'_n$ and $\lambda := \sum a_n \mu_n \in \mathbf{R}_{++}$ to conclude that $\{x \in Y : |\langle x, x'\rangle| \leqslant \lambda\} \sqsubseteq U$ which finishes the proof. □

Definition 4.50. Let $\langle X, X'\rangle$ be a conditional dual pair and $Y \in \mathcal{S}(X)$. Then the conditional polar and one-sided polar of Y is defined as

$$Y^\bullet := \{x' \in X' : |\langle x, x'\rangle| \leqslant 1 \text{ for all } x \in Y\};$$
$$Y^\circ := \{x' \in X' : \langle x, x'\rangle \leqslant 1 \text{ for all } x \in Y\};$$

respectively.

Note that both $Y^\bullet$ and Y° are in $\mathcal{S}(X')$ as they both contain 0.

Lemma 4.51. *Let $\langle X, X'\rangle$ be a conditional dual pair, $(Y_i) \sqsubseteq \mathcal{S}(X)$ and $Y, Z \in S(X)$. Then the following holds:*

(i) If $Z \sqsubseteq Y$, then $Y^\bullet \sqsubseteq Z^\bullet$ and $Y^\circ \sqsubseteq Z^\circ$.

(ii) If $\lambda \in \{0\}^\sqsubset$, then $(\lambda Y)^\bullet = (1/\lambda)Y^\bullet$ and $(\lambda Y)^\circ = (1/\lambda)Y^\circ$.

(iii) $(\sqcup Y_i)^\bullet = \sqcap Y_i^\bullet$ and $(\sqcup Y_i)^\circ = \sqcap Y_i^\circ$.

(iv) $Y^\bullet$ is conditionally convex, $\sigma(X', X)$-closed, circled and contains 0.

(v) Y° is conditionally convex, $\sigma(X', X)$-closed and contains 0.

(vi) If Y is conditionally absorbing, then both $Y^\bullet$ and Y° are conditionally $\sigma(X', X)$-bounded.

(vii) Y is conditionally $\sigma(X, X')$-bounded if and only if $Y^\bullet$ is conditionally absorbing.

Proof. The first two properties are a direct consequence of the definition. To prove *(iii)*, we first obtain that $\sqcap Y_i^\bullet = \cap Y_i^\bullet$ and $\sqcap Y_i^\circ = \cap Y_i^\circ$, since $Y_i^\bullet$ and Y_i° are elements of $S(X')$. It holds that $\cap_{i\in I} Y_i^\bullet = \{x' \in X' : |\langle x, x'\rangle| \leqslant 1 \text{ for all } x \in Y_i, i \in I\}$. By definition of the conditional union, it holds

$$(\sqcup_{i\in I} Y_i)^\bullet = \left\{x' \in X' : \left|\langle \sum a_n x_{i_n}, x'\rangle\right| \leqslant 1 \text{ for all } (a_n) \in \mathcal{K}(1), x_{i_n} \in Y_{i_n}, i_n \in I\right\}.$$

Choosing a partition $(a_n) = \{1\}$ in the set on the right-hand-side, we obtain $(\sqcup Y_i)^\bullet \sqsubseteq \cap Y_i^\bullet$. Reversely, since $|.|$ and $\langle ., x'\rangle$ are conditional functions, it holds $|\langle \sum a_n x_{i_n}, x'\rangle| = \sum a_n |\langle x_{i_n}, x'\rangle|$. Moreover, if $a_n |\langle x_{i_n}, x'\rangle| \leqslant a_n 1$ it follows that $|\langle a_n x_{i_n} + a_n^c 0, x'\rangle| \leqslant 1$ which shows the equality of both sets. Concerning *(iv)* the properties of being conditionally convex and circled follow immediately. To show the conditional closedness consider a conditional net $(x'_\alpha) \subseteq Y^\bullet$ conditionally converging to x' with respect to

$\sigma(X', X)$. By definition, it holds for all α that $|\langle x, x'_\alpha\rangle| \leqslant 1$ for every $x \in Y$. Moreover, $\langle x, .\rangle$ is by definition conditionally continuous with respect to $\sigma(X', X)$. Therefore, it holds $|\langle x, x'\rangle| = \lim |\langle x, x'_\alpha\rangle| \leqslant 1$ for every $x \in Y$ which shows $x' \in Y^\bullet$ and hence $Y^\bullet$ is conditionally closed with respect to $\sigma(X', X)$. The claim for Y° follows analogously. To show *(vi)* let Y be conditionally absorbing and fix $x \in Y$. This means, that there exists an $\lambda_x \in \mathbf{R}_{++}$ such that both $\lambda_x x$ and $-\lambda_x x$ are in Y. Hence, for any $x' \in Y^\bullet$ it holds $|\langle \lambda_x x, x'\rangle| \leqslant 1$ by definition. For any $y' \in Y^\circ$ it holds both $\langle \lambda_x x, y'\rangle \leqslant 1$ and $\langle -\lambda_x x, y'\rangle \leqslant 1$ which yields $|\langle \lambda_x x, y'\rangle| \leqslant 1$. By conditional linearity of $\langle ., x'\rangle$, and since $\lambda_x \in \{0\}^\sqsubset$, we can conclude that $|\langle x, x'\rangle| \leqslant 1/\lambda_x$ for every $x' \in Y^\bullet$ and $|\langle x, y'\rangle| \leqslant 1/\lambda_x$ for every $y' \in Y^\circ$. In line with Lemma 4.49 this shows that both $Y^\bullet$ and Y° are conditionally bounded with respect to $\sigma(X', X)$ To prove *(vii)* suppose Y is conditionally bounded with respect to $\sigma(X', X)$ and fix $x' \in Y^\bullet$. By Lemma 4.49, there exists $\lambda_{x'} \in \mathbf{R}_{++}$ such that $|\langle x, x'\rangle| \leqslant 1/\lambda_{x'}$ for any $x \in Y$. By conditional linearity of $\langle x, .\rangle$, and since $\lambda_{x'} \in \{0\}^\sqsubset$, it follows that $|\langle x, \lambda x'\rangle| \leqslant 1$ for each $x \in Y$, and thereby $\lambda x' \in Y^\bullet$ for any $0 \leqslant \lambda \leqslant \lambda_{x'}$. Thus, $Y^\bullet$ is conditionally absorbing. For the second implication, let $Y^\bullet$ be conditionally absorbing. By *(vi)* and Corollary 4.48, it follows that $Y^{\bullet\bullet}$ is conditionally $\sigma(X, X')$-bounded. Since $Y \sqsubseteq Y^{\bullet\bullet}$, it follows that Y is also conditionally $\sigma(X, X')$-bounded. Indeed, $Y^{\bullet\bullet} := \{x \in X : |\langle x, x'\rangle| \leqslant 1 \text{ for all } x' \in Y^\bullet\}$. By definition of $Y^\bullet$, we compare $Y \sqsubseteq Y^{\bullet\bullet}$. □

Using these properties we can prove the conditional version of the Bipolar Theorem. Therein the term smallest is meant with respect to the conditional inclusion.

Theorem 4.52. *Let $\langle X, X'\rangle$ be a conditional dual pair and $Y \in \mathcal{S}(X)$. Then it holds that*

(i) *$Y^{\bullet\bullet}$ is the smallest conditionally convex, circled, $\sigma(X, X')$-closed set in $S(X)$ conditionally including Y. Therefore, if Y is conditionally convex, circled and $\sigma(X, X')$-closed it holds that $Y^{\bullet\bullet} = Y$.*

(ii) *$Y^{\circ\circ}$ is the smallest conditionally convex, $\sigma(X, X')$-closed set in $S(X)$ conditionally including $Y \sqcup \{0\}$. Thus, if Y is conditionally convex, $\sigma(X, X^\star)$-closed and contains 0 it holds that $Y^{\circ\circ} = Y$.*

Proof. We define

$$\tilde{Y} = \sqcap \{Z \in \mathcal{S}(X) : Y \sqsubseteq Z, Z \text{ is conditionally convex, circled and } \sigma(X, X')\text{-closed}\},$$

which is conditionally convex, circled and $\sigma(X, X')$-closed as these properties are stable under conditional intersection. Due to property *(iv)* in Lemma 4.51 and Corollary 4.48 , it holds that $Y^{\bullet\bullet}$ is conditionally convex, circled, and $\sigma(X, X')$-closed and hence $Y \sqsubseteq \tilde{Y} \sqsubseteq Y^{\bullet\bullet}$. Suppose now that $Y^{\bullet\bullet} \sqcap \tilde{Y} \neq \mathbf{0}$. This conditional intersection is in $\mathcal{S}(aX)$ for some $a > 0$, and without loss of generality, we assume $a = 1$. Pick $x \in X$

with $x \in Y^{\bullet\bullet} \sqcap \tilde{Y}^{\sqsubset}$. It holds that $\{x\}$ is a conditionally compact, convex set and $\tilde{Y}$ is conditionally $\sigma(X, X')$-closed and convex. Hence, by means of Theorem 4.44 and Corollary 4.48, there exists $x' \in X'$ and $t \in \mathbf{R}_{++}$ such that $|\langle \tilde{x}, x' \rangle| \leqslant t < \langle x, x' \rangle$ for all $\tilde{x} \in \tilde{Y}$, where we used that Y is conditionally circled which implies that for $x \in Y$ also $-x \in Y$. Up to rescaling by $1/t$, we may assume that $t = 1$, and thus $|\langle \tilde{x}, x' \rangle| \leqslant 1$ for all $\tilde{x} \in \tilde{Y}$. Since $Y \sqsubseteq \tilde{Y}$, it follows that $x' \in Y^{\bullet}$. However, $\langle x', x \rangle > 1$ contradicts the fact that $x \in Y^{\bullet\bullet}$.

For the second assertion we define

$$\hat{Y} = \sqcap \{ Z \in \mathcal{S}(X) : Y \sqcup \{0\} \sqsubseteq Z, Z \text{ is conditionally convex and } \sigma(X, X')\text{-closed} \},$$

which is conditionally convex, $\sigma(X, X')$-closed and contains $Y \sqcup \{0\}$. Due to Due to property *(v)* in Lemma 4.51 and Corollary 4.48 , it holds that also $Y^{\circ\circ}$ has this properties which shows $Y \sqcup \{0\} \sqsubseteq \hat{Y} \sqsubseteq Y^{\circ\circ}$. The rest of the proof follows the same argumentation as above. That is for $x \in X$ with $x \in Y^{\circ\circ} \sqcap \hat{Y}^{\sqsubset}$ we can apply Theorem 4.44 and Corollary 4.48 to $\{x\}$ and $\hat{Y}$. Thus there exists $x' \in X'$ and $t \in \mathbf{R}_{++}$ such that $\langle \hat{x}, x' \rangle \leqslant t < \langle x, x' \rangle$ for all $\hat{x} \in \hat{Y}$. Up to rescaling by $1/t$, we may assume that $t = 1$, and thus $\langle \hat{x}, x' \rangle \leqslant 1$ for all $\hat{x} \in \hat{Y}$. Since $Y \sqsubseteq \hat{Y}$, it follows that $x' \in Y^{\circ}$. However, $\langle x', x \rangle > 1$ contradicts the fact that $x \in Y^{\circ\circ}$. □

Next we prove the conditional version of the Banach-Alaoglu Theorem.

Theorem 4.53. *Let $\langle X, X' \rangle$ be a conditional dual pair where X is a conditional locally convex topological vector space with a conditional topology which is compatible with $\langle X, X' \rangle$. Then if U is a conditional neighborhood of 0 it holds that both $U^{\bullet}$ and U° are conditionally $\sigma(X', X)$-compact.*

Proof. The conditional $\sigma(X', X)$-topology is the conditional topology of conditional pointwise convergence in X and hence induced by the conditional product topology on $\mathcal{M}(X, \mathbf{R})$. By the conditional version of Tychonoff's Theorem 4.28, it is sufficient to show that $U_x^{\bullet} = \langle x, U^{\bullet} \rangle := \{ \langle x, x' \rangle : x' \in U^{\bullet} \} \sqsubseteq \mathbf{R}$ and $U_x^{\circ} = \langle x, U^{\circ} \rangle := \{ \langle x, x' \rangle : x' \in U^{\circ} \} \sqsubseteq \mathbf{R}$ are conditionally compact, respectively, for every $x \in X$. We will do the proof for $U^{\bullet}$ and the proof for U° follows analogously. By the conditional Heine-Borel Theorem 4.32, it suffices to show that $U_x^{\bullet}$ is conditionally closed and bounded in $\mathbf{R}$. To show the conditional closedness let U be a conditional neighborhood of 0. Let $(x'_\alpha) \subseteq U^{\bullet}$ be a conditional net conditionally converging to some $x' \in X'$. By conditional continuity of $\langle x, . \rangle$, it follows that $|\langle x, x' \rangle| = \lim |\langle x, x'_\alpha \rangle| \leqslant 1$ for every $x \in U$, and therefore $x' \in U^{\bullet}$ showing that $U_x^{\bullet}$ is conditionally closed by the characterization in Proposition 4.24. To show the conditional boundedness consider $x \in X$ and let U be a conditional neighborhood of 0. Since X is a ctvs, there exists $t_x \in \mathbf{R}_{++}$ such that $(1/t_x)x \in U$. Hence, $|\langle x, x' \rangle| \leqslant t_x$ for every $x' \in U^{\bullet}$ showing that $U_x^{\bullet}$ is conditionally bounded in $\mathbf{R}$ for every $x \in X$. □

4.2.3 Normed Vector Spaces

Definition 4.54. Let X be a conditional vector space. A conditional norm is a conditional function $\|\cdot\| : X \to \mathbf{R}_+$ such that

(i) $\|x\| = 0$ if and only if $x = 0$;

(ii) $\|\lambda x\| = |\lambda| \, \|x\|$ for every $x \in X$ and $\lambda \in \mathbf{R}$;

(iii) $\|x + y\| \leqslant \|x\| + \|y\|$ for every $x, y \in X$.

A conditional vector space together with a conditional norm is called a conditional normed vector space. For $x \in X$ and $\varepsilon \in \mathbf{R}_{++}$, we denote by $B_\varepsilon(x) = \{y \in X : \|x - y\| < \varepsilon\}$ a conditionally open ball. We can define the conditional topology $\mathcal{T}$ generated by the collection $\{B_\varepsilon(x) : x \in X, \varepsilon \in \mathbf{R}_{++}\}$ called the conditional topology induced by the conditional norm.

A set $Z \in \mathcal{S}(X)$ in a conditional normed space $(X, \|.\|)$ is called conditionally norm-bounded if there exists $M \in \mathbf{R}_+$ such that $\|x\| \leqslant M$ for every $x \in Z$.

Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be conditional normed vector spaces. Then for a conditionally linear operator $T : X \to Y$ we define the conditional operator norm by

$$\|T\|^\star := \sup \{\|T(x)\| : x \in X, \|x\| = 1\}.$$

If $\|T\|^\star \in \mathbf{R}_+$ we call T conditionally norm-bounded.

Lemma 4.55. *Let $T : X \to Y$ be a conditionally linear operator between two conditional normed vector spaces $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$. Then, T is conditionally continuous if and only if T is conditionally norm-bounded.*

Proof. First we notice that T is conditionally norm-bounded if and only if $\|T(x)\| \leqslant r \|x\|$ for all $x \in X$ and some $r \in \mathbf{R}_+$. This is due to the identity $\|T\|^\star = \inf M$ where

$$M = \{r \in \mathbf{R}_+ : \|T(x)\| \leqslant r \|x\| \text{ for all } x \in X\}.$$

To see this denote by $r_0 = \inf M$. By definition, $ar_0 = a0$ if and only if $a \|T\|^\star = a0$ so suppose without loss of generality that both are strictly greater then 0. Further it holds

$$\left\| T\left(\frac{x}{\|x\|}\right) \right\| = \frac{1}{\|x\|} \|T(x))\|,$$

for $x \in 0^\sqsubset$, since T is conditionally linear. Therefore, if $\|T\|^\star > 0$ it holds that

$$\|T\|^\star = \sup \left\{ \frac{1}{\|x\|} \|T(x))\| : x \in X \right\},$$

showing $\|T\|^\star \in M$ and thus, $r_0 \leqslant \|T\|^\star$. By the definition of the conditional supremum, for any $\varepsilon \in \mathbf{R}_{++}$ there exists $x_\varepsilon \in X$ with $\|T(x_\varepsilon)\| \geqslant \|T\|^\star (1 - \varepsilon) \|x_\varepsilon\|$. Therefore, for any $s \in M$ it has to hold $s \geqslant \|T\|^\star (1-\varepsilon)$ which gives $r_0 \geqslant \|T\|^\star (1-\varepsilon)$. Hence, $r_0 = \|T\|^\star$. Thus, if T is conditionally bounded and linear, there exists some $r \in \mathbf{R}$ such that for all $x_n, x_0 \in X$ it holds

$$\|T(x_n) - T(x_0)\| = \|T(x_n - x_0)\| \leqslant r \|x_n - x_0\|.$$

Therefore, for every conditional sequence (x_n) where $\|x_n - x_0\|$ conditionally converges to zero also $\|T(x_n) - T(x_0)\|$ does so. Thus, T is conditionally continuous.

If T is conditionally continuous, then in particular at 0. Assume now that T is not conditionally bounded. Define $b = \vee\{a : a\|T\|^\star \notin a\mathbf{R}_+\}$. Hence, there exists a conditional sequence (x_n) such that $b\|T(x_n)\|_Y > bn\|x_n\|$ for any n. Since this inequality is conditionally strict, it holds that $ax_n \neq a0$ for all $a \leq b$. Otherwise, because the conditional linearity of T yields $T(0) = 0$, we would obtain an equality on a contradicting the conditional strictness.

Thus we can define $y_n = b(1/n\|x_n\|)x_n + b^c 1/n$. It follows that $\|y_n\| = 1/n$ for any n. Thus (y_n) conditionally converges to 0. Moreover, $b\|T(y_n)\| = b(1/n\|x_n\|)\|T(x_n)\| > b1$ which contradicts that $(T(y_n))$ conditionally converges to $T(0) = 0$. Thus, T would not be conditionally continuous at 0 which is a contradiction. □

For a conditional norm $\|\cdot\| : X \to \mathbf{R}$ the conditional function $d : X \times X \to \mathbf{R}$ defined by $d(x, y) = \|x - y\|$ is a conditional metric with $d(\lambda x, \lambda y) = |\lambda| d(x, y)$.

Definition 4.56. A conditional normed vector space X is called a conditional Banach space if X is conditionally complete for the conditional metric $d(x, y) = \|x - y\|$.

We will close this chapter by showing the conditional version of the Krein-Šmulian Theorem.

In the following theorem we will use the conditional topology $\sigma(X^\star, X)$ for the conditional dual space $X^\star$ of X adapting the argumentation in [41]. For a proof for this theorem in L^∞-modules compare [38]. Since X and its conditional dual space $X^\star$ form a conditional dual pair, we can use $\sigma(X^\star, X)$. Further, for $\varepsilon \in \mathbf{R}_{++}$ we denote $C^\star_\varepsilon(0) = \{y^\star \in X^\star : \|y\|^\star \leqslant \varepsilon\}$ and $C_\varepsilon(x) := \{y \in X : \|x - y\| \leq \varepsilon\}$.

Theorem 4.57. *Let X be a conditional Banach space and $X^\star$ its conditional dual space. Let $A \in \mathcal{S}(X^\star)$ be a conditionally convex set. Then $A \sqcap C^\star_n(0)$ is conditionally $\sigma(X^\star, X)$-closed for every $n \in \mathbf{N}$ if and only if A is conditionally $\sigma(X^\star, X)$-closed.*

Proof. If A is conditionally $\sigma(X^\star, X)$-closed then its conditional intersection with $C^\star_n(0)$ is so, since $C^\star_n(0)$ is also a conditionally $\sigma(X^\star, X)$-closed set.

To show the reverse implication, we suppose that $A \sqcap C^\star_n(0)$ is conditionally $\sigma(X^\star, X)$-closed for every $n \in \mathbf{N}$. From now on we denote by $C := C_1(0)$ and $C^\star := C^\star_1(0)$. Note first that A is conditionally norm-closed. Indeed, for a conditional sequence $(x^\star_n) \subseteq A$ such that $x^\star_n \to x^\star$, it follows that $(x^\star_n) \subseteq A \sqcap C^\star_m$ for $m \in \mathbf{N}$ large enough. However, conditional norm-convergence implies conditional $\sigma(X^\star, X)$-convergence, [3] and since $x^\star_n \in A \sqcap C^\star_m$ which is conditionally $\sigma(X^\star, X)$-closed by assumption, it follows that $x^\star \in A$. Up to conditional addition of A and a single element, which is a conditionally continuous operation, we may assume that $0 \in A$ and define $A_n := A \sqcap 2^n C^\star$ for every $n \in \mathbf{N}$. Every A_n is conditionally $\sigma(X^\star, X)$-closed by assumption, and conditionally convex , since $C^\star_n$ is so according to Theorem 4.52 and A is conditionally convex by assumption. Moreover, every A_n contains zero by the previous explanation and hence it follows that $A_n \in \mathcal{S}(X^\star)$. Furthermore, since the conditional closed balls are $\mathcal{A}$-stable, it follows that (A_n) is a conditional family such that $A = \sqcup A_n = \cup A_n$. By the conditional Bipolar Theorem 4.52, it holds $A^{\circ\circ}_n = A_n$. We define $B = \sqcap A^\circ_n = \cap A^\circ_n$ and will show that $A = B^\circ$, and therefore A is conditionally $\sigma(X^\star, X)$-closed, compare Theorem 4.52. On the one hand, $B \sqsubseteq A^\circ_n$ for every n, hence $A_n \sqsubseteq B^\circ$ for every n, and therefore $A = \sqcup A_n \sqsubseteq B^\circ$. Let us show the reverse conditional inclusion.

Step 1: We show that $A^\circ_n \sqsubseteq A^\circ_{n+1} + 2^{-n}C$ for every $n \in \mathbf{N}$. Let $x \in \left(A^\circ_{n+1} + 2^{-n}C\right)^\sqsubset$. Without loss of generality, we may assume that $\left(A^\circ_{n+1} + 2^{-n}C\right)^\sqsubset \in \mathcal{S}(X)$. By Lemma 4.39*(v)* and Theorem 4.53 [4], we may apply Theorem 4.44 and get $x^\star \in X^\star$ such that

$$\langle x, x^\star\rangle \geqslant 1 \geqslant \sup\langle A^\circ_{n+1} + 2^{-n}C, x^\star\rangle = \sup\langle A^\circ_{n+1}, x^\star\rangle + 2^{-n}\left\|x^\star\right\|,$$

showing that $\sup\langle A^\circ_{n+1}, x^\star\rangle \leqslant 1 - 2^{-n}\|x^\star\|$. Further, from $A_n \sqsubseteq 2^{n+1}C^\star$, it follows $2^{-(n+1)}C \sqsubseteq A^\circ_n$, and $\langle x^\star, 3\cdot 2^{-(n+1)}C\rangle = \langle x^\star, 2^{-(n+1)}C + 2^{-n}C\rangle \sqsubseteq \langle x^\star, A^\circ_{n+1} + 2^{-n}C\rangle$, yields $\|x^\star\| \leqslant 2^{n+1}/3$. For $0 < \varepsilon \leqslant 1/3 \wedge (2^{-n}\left\|x^\star\right\|)$, it follows from the previous inequalities that $y^\star = x^\star/(1-\varepsilon)$ fulfills $\sup\langle A^\circ_{n+1}, y^\star\rangle \leqslant 1$, that is $y^\star \in A^{\circ\circ}_{n+1} = A_{n+1}$, and $\|y^\star\| = \left\|x^\star\right\|/(1-\varepsilon) \leqslant 2^n$ showing that $y^\star \in A_n$. However, $\langle x, y^\star\rangle > 1$ which implies that $x \in (A^\circ_n)^\sqsubset$. [5]

Step 2: We show that $A^\circ_n \sqsubseteq B + 2^{-(n-1)}C$ for every $n \in \mathbf{N}$. For $k \in N$, we choose recursively according to the previous step $x^n_k \in A^\circ_{n+k}$ such that $\left\|x^n_k - x^n_{k+1}\right\| \leqslant 2^{-(n+k)}$. Since the families we consider are conditional, it follows that $x^n_k = \sum a_i x^n_{k_i}$ where $k = \sum a_i k_i \in \mathbf{N}$, defines a conditional sequence such that $x^n_k \in \sum a_i A^\circ_{n+k_i} = A^\circ_{n+k}$ and $\|x^n_k - x^n_{k'}\| \leqslant 2^{-(n+k-1)}(1-2^{k-k'})$. It follows that (x^n_k) is a conditional Cauchy sequence in X, and by conditional completeness of X it follows $x^n_k \to x^n$ for some $x^n \in X$. However, A°_n is conditionally norm-closed, since it is conditionally $\sigma(X, X^\star)$-closed and conditionally norm-bounded. Hence, $x^n \in \sqcap A^\circ_{n+k} = B$. From $\|x^n_0 - x^n\| \leqslant 2^{1-n}$ and

[3] In a conditional Banach space X, conditional norm-convergence implies conditional $\sigma(X, X^\star)$-convergence. Indeed, let $(x_n) \subseteq X$ be a conditionally norm-converging sequence to $x \in X$ and $x^\star \in X^\star$. It holds $\|\langle x_n, x^\star\rangle - \langle x, x^\star\rangle\| \leqslant \|x^\star\|\,\|x_n - x\| \to 0$, compare Lemma 4.55 for the inequality. Hence, (x_n) conditionally converges to x in the conditional $\sigma(X, X^\star)$-topology.

[4] Indeed, A°_{n+1} is conditionally $\sigma(X, X^\star)$-closed and $2^{-n}C^\star$ is conditionally $\sigma(X, X^\star)$-compact by Theorem 4.53, it follows that $A^\circ_{n+1} + 2^{-n}C^\star$ is conditionally closed, compare Lemma 4.39 *(v)*, and $\{x\}$ is clearly conditionally $\sigma(X, X^\star)$-compact.

[5] Since $a\langle x, y^\star\rangle > a1$ for every $a \in \mathcal{A}\cdot a > 0$ so that $x \notin aA^\circ_n$ for every $a > 0$.

$x^n \in G$ follows that $x_0^n \in B + 2^{-(n-1)}C$ for every $x_0^n \in A_n^\circ$.

Step 3: We show that $A = \sqcap_{\varepsilon\in\mathbf{R}_{++}}(1+\varepsilon)A$. Since A is conditionally convex and $0 \in A$, the conditional inclusion $\sqsubseteq$ is immediate. Conversely, for $x^\star \in \sqcap_{\varepsilon\in\mathbf{R}_{++}}(1+\varepsilon)A$, setting $x_n^\star = n/(1+n)x^\star$ for $n \in \mathbf{N}$ defines a conditional sequence in A such that $\|x^\star - x_n^\star\| \leqslant \|x^\star\|/(n+1) \to 0$. Hence, $x_n \to x$, and since A is norm-closed, it follows that $x \in A$.

Step 4: We show that $B^\circ \sqsubseteq \sqcap_{\varepsilon\in\mathbf{R}_{++}}(1+\varepsilon)A$. Given $\varepsilon \in \mathbf{R}_{++}$, it holds $x + y = (1+\varepsilon)(\frac{x}{1+\varepsilon} + (1 - \frac{1}{1+\varepsilon})\frac{y}{\varepsilon})$ for every $x, y \in X$. From the second step, it follows that $A_n^\circ \sqsubseteq (1+\varepsilon)\operatorname{conv}(B \sqcup 2^{1-n}/\varepsilon C)$. Hence, taking the conditional polars, it follows that $1/(1+\varepsilon)(B^\circ \sqcap 2^{n-1}\varepsilon C^\star) \sqsubseteq A_n \sqsubseteq A$ for every $n \in \mathbf{N}$. Finally, taking the conditional union over $n \in \mathbf{N}$, yields $1/(1+\varepsilon)B^\circ \sqsubseteq A$ for every $\varepsilon \in \mathbf{R}_{++}$, and so $B^\circ \sqsubseteq \sqcap_{\varepsilon\in\mathbf{R}_{++}}(1+\varepsilon)A = A$ by means of the third step.

□

5 Transitivity

Throughout this chapter, $\mathcal{X}$ denotes a set of elements and $\succsim$ a binary relation defined on it. We always identify the relation with the set $R = \{(x, y) : x \succsim y\} \subseteq \mathcal{X} \times \mathcal{X}$. Further, we denote the upper level set of x by $U(x) = \{y \in \mathcal{X} : y \succsim x\}$. We will use $\not\succsim$ to denote that $\succsim$ is not fulfilled. The strict better part $\succ$ of a relation $\succsim$ is defined as follows: $x \succ y$ if and only $x \succsim y$ and $y \not\succsim x$. Moreover, for $x \succsim y$, $y \succsim z$ we write for short $x \succsim y \succsim z$, and $x, y \succsim z$ denotes $x \succsim z$ and $y \succsim z$. A relation is reflexive if $x \succsim x$ for every $x \in \mathcal{X}$ and transitive if $x \succsim y \succsim z$ implies $x \succsim z$, for $x, y, z \in \mathcal{X}$.

5.1 Representations of Relations

In this section, we summarize ways of representing relations. Since the first half of the 20th century, mathematicians and economists, among others Debreu, Fishburn, von Neumann and Savage, have worked on the subject of representations of relations. The form of representation is determined by the properties of the examined relation and these properties are often considered by the authors to be normatively relevant.

Numerical Representation

The basic type of representing a relation is called numerical representation. A single function, called utility function, $u : \mathcal{X} \to \mathbb{R}$ represents the relation by means of

$$x \succsim y \quad \Longleftrightarrow \quad u(x) \geq u(y).$$

To obtain such a representation the relation has to be complete, transitive and to have a countable order dense subset (compare [49, Theorem 2.6]).

Multi-Utility Representation (MU)

The multi-utility representation (MU) (compare [40]) is not based on a single function but on a family $\mathcal{U} = (u_i)_{i \in I}$ of functions $u_i : \mathcal{X} \to \mathbb{R}$, $i \in I$, where I is an arbitrary index set. The relation is represented by means of

$$x \succsim y \quad \Longleftrightarrow \quad u_i(x) \geq u_i(y), \text{ for all } i \in I.$$

To obtain an MU, the properties of transitivity and reflexivity are sufficient and necessary. Indeed, denoting by 1_A the indicator function of a set A, the collection $(1_{U(x)})_{x\in\mathcal{X}}$ is an MU. Any utility function u_i represents a complete relation as a numerical representation. Thus, the approach amounts to writing $\succsim$ as the intersection of complete relations all allowing for a numerical representation, and moreover sharing further properties of $\succsim$.

Richter-Peleg-Representation

The Richter-Peleg representation of a relation is of the form

$$\begin{aligned} x \succ y \quad &\Longrightarrow \quad f(x) > f(y), \\ x \sim y \quad &\Longrightarrow \quad f(x) = f(y), \end{aligned}$$

for a function $f : \mathcal{X} \to \mathbb{R}$. Since f can be considered as the numerical representation of some relation $\succsim_1$, finding a Richter-Peleg representation is equivalent to the following question. Given a relation $\succsim$, does there exist an extending relation $\succsim_1$ allowing for a numerical representation? That $\succsim_1$ extends $\succsim$ means that $x \succsim y$ implies $x \succsim_1 y$ and $x \succ y$ implies $x \succ_1 y$, meaning the strict better part of $\succsim$ is also contained in the one of $\succsim_1$. A Richter-Peleg representation is very suitable for maximization purposes, however f does not characterize the entire relation.

Bivariate Representation

A representation tailored for nontransitive relations is the bivariate representation (compare [44, 46]), which is given by a function $\phi : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ in terms of

$$x \succsim y \quad \Longleftrightarrow \quad \phi(x, y) \geq 0,$$

or, respectively, in terms of

$$x \succ y \quad \Longleftrightarrow \quad \phi(x, y) > 0. \tag{5.1}$$

Without demanding further properties of ϕ this is trivial to obtain. Indeed, one can simply define ϕ by

$$\phi(x, y) = \begin{cases} 1, & \text{if } x \succ y, \\ 0, & \text{if } x \sim y, \\ -1, & \text{if } x \not\succsim y, \end{cases}$$

to represent $\succsim$ in both ways. Therefore, the so called SSB-representation of the form (5.1) of relations on vector spaces was introduced. The function ϕ was demanded to be skew-symmetric, that is $\phi(x, y) = -\phi(y, x)$, for all $x, y \in \mathcal{X}$, and bilinear, SSB for short.

For results on SSB-representations compare [46]) and the references therein.

Set-valued Representation

The concept of a set-valued representation, also called containment representation (compare [47]), goes back to [35], though not studied in general terms there. The approach consists in finding a space $\mathcal{Y}$ and a set-valued function $f : \mathcal{X} \to \mathcal{P}(\mathcal{Y})$, where $\mathcal{P}(\mathcal{Y})$ denotes the power set of $\mathcal{Y}$, such that

$$x \succsim y \quad \Longleftrightarrow \quad f(x) \subseteq f(y).$$

Any set-valued represented relation is transitive and reflexive, since the inclusion relation is so. Reversely, any transitive, reflexive relation has a set-valued representation. Indeed, the function $f : x \mapsto U(x)$ mapping to each element its upper level set is a set-valued representation. A special form of set-valued representation is the field of interval orders (compare [45]).

Threshold-Representation

Starting with the paper [62], the theory of the threshold-representation was developed. The aim is to find a utility function $u : \mathcal{X} \to \mathbb{R}$ and a nonnegative function $t : \mathcal{X} \to \mathbb{R}_{++}$ such that

$$x \succ y \quad \Longleftrightarrow \quad u(x) > u(y) + t(y).$$

The function t can be seen as a penalty function. Note that a relation represented in this way has to be transitive.

Relations Induced by Cones

Let $\mathcal{X}$ be a vector space. A relation is said to be induced by a closed convex cone C if

$$x \succsim y \quad \Longleftrightarrow \quad x - y \in C.$$

Such a relation is reflexive, transitive and stable under linear transformation.

Relations Induced by Moving Sets

In a vector space $\mathcal{X}$, a relation is said to be induced by the moving sets $(D(x))_{x \in \mathcal{X}}$ or to have a variable ordering structure if

$$x \succsim y \quad \Longleftrightarrow \quad x - y \in D(y).$$

Thus, the term “moving” expresses that there is no single globally fixed set inducing the relation but multiple ones. Such a relation is reflexive if and only if every $D(x), x \in \mathcal{X}$, contains zero. The transitivity can be characterized as follows.

Lemma 5.1. *For a reflexive relation $\succsim$ induced by moving sets $(D(x))_{x\in\mathcal{X}}$, the following are equivalent.*

(i) *The relation is transitive.*

(ii) *It holds that $y \succsim z$ if and only if $y + D(y) \subseteq z + D(z)$.*

Proof. Suppose first the relation to be transitive. If $y \succsim z$, then $x \succsim y$ always implies $x \succsim z$, for any x. Hence, $x \in y + D(y)$ always implies $x \in z + D(z)$, showing $y + D(y) \subseteq z + D(z)$. If $y + D(y) \subseteq z + D(z)$, then $D(y)$ containing zero implies in particular that $y \in z + D(z)$, showing $y \succsim z$.

For the reverse implication, suppose that $y \succsim z$ always implies $y + D(y) \subseteq z + D(z)$. Consider a chain $x \succsim y \succsim z$. Specifically, it holds that $x \in y + D(y)$ and since $y \succsim z$, it follows that $x \in z + D(z)$, showing $x \succsim z$. ☐

5.2 Relations Induced by Moving Convex Cones

Throughout this section, $\mathcal{X}$ denotes a locally convex topological vector space. We consider representations of the form $x \succsim y$ if and only if $x - y \in C(y)$, where $(C(x))_{x\in\mathcal{X}}$ are convex cones in $\mathcal{X}$ containing zero. As $U(x) = C(x) + x$, it follows that every upper level set $U(x)$ is convex. The case of constant sets, meaning $C(x) = C$ for all $x \in \mathcal{X}$, will be denoted by "the relation is induced by a convex cone" and the case that the sets $C(x)$ depend on x will be expressed by the term" the relation is induced by moving convex cones". For further details of representations of that form compare [36, 37] and the references therein.

We start with the case that the relation is induced by a convex cone $C \subseteq \mathcal{X}$ containing zero which causes the relation to be reflexive and transitive. Moreover, the relation is complete if and only if $C \cup (-C) = \mathcal{X}$ and antisymmetric if and only if $C \cap (-C) = \{0\}$. With methods of convex analysis one can show that a relation induced by a closed, convex cone is strongly connected to an MU. Recall that a cone is called polyhedral if it is the intersection of finitely many half-spaces.

Lemma 5.2. *A relation induced by a closed, convex cone has an MU consisting of continuous and linear functions. Reversely, if a relation on a vector space is represented by an MU of continuous and linear functions, it is also induced by a closed, convex cone. In particular, the MU is finite if and only if the cone is polyhedral or $C = \mathcal{X}$.*

Proof. It holds that $C = \mathcal{X}$ if and only if $x \succsim y$ for every $x, y \in \mathcal{X}$. The utility function $u(x) = 0$, $x \in \mathcal{X}$ represents this relation. If C is a closed, convex cone with $C \neq \mathcal{X}$, it is the intersection of all half-spaces containing it (compare [3, Corollary 5.83]). These half-spaces are of the form $H(x^*) = \{x : x^*(x) \geq 0\}$ for $x^* \in \mathcal{X}^*$. It therefore holds that $x - y \in C$ if and only if $x^*(x - y) \geq 0$ for all $x^* \in \mathcal{X}^*$. Since every x^* is linear, this is equivalent to $x^*(x) \geq x^*(y)$ for all $x^* \in \mathcal{X}^*$ and hence $(x^*)_{x^* \in \mathcal{X}^*}$ is an MU of the relation. If further C is polyhedral, then there exists finitely many half-spaces whose intersection is C, and hence we obtain a finite MU.

Reversely, suppose $x \succsim y$ if and only if $u_i(x) \geq u_i(y)$ for all i in some index set I, where all functions are continuous and linear. Hence, it follows that $x \succsim y$ if and only if $u_i(x - y) \geq 0$, for all $i \in I$. Therefore, every $H_i = \{x \in \mathcal{X} : u_i(x) \geq 0\}$, $i \in I$, defines a half-space. Defining the closed, convex cone $C = \cap_{i \in I} H_i$, it holds that $x \succsim y$ if and only if $x - y \in C$. In the case that I is finite, C is the intersection of finitely many half-spaces and hence a polyhedral cone. $\square$

We now examine relations induced by moving convex cones and start by exemplifying such a relation.

Example 5.3. Walking through a dark area with a flashlight, the battery of the flashlight is getting weaker with every minute and hence the light cone is getting smaller. Starting at a certain point, one is walking in the direction of the light cone and the light cone at the next point remains in the old one. In that way, one walks from x_n to x_{n+1} with $x_{n+1} \in x_n + C_n$ where C_n is a cone and $C_{n+1} \subseteq C_n$ for every n. The fact that the battery is dead at time T corresponds to $C_T = \{0\}$. This results in the chain $x_1 \precsim x_n \precsim \cdots \precsim x_T$ for the relation induced by moving cones. The property $C_{n+1} \subseteq C_n$ causes the relation to be transitive and hence it also holds that $x_m \succsim x_k$ for every $m \geq k$. Thus, x_T is the maximal element of this chain.

In the next lemma we use the polar cone operator which for a set D is defined as $D^\circ := \{x^* \in \mathcal{X}^* : x^*(x) \geq 0, \forall x \in D\}$.

Lemma 5.4. *For a relation $\succsim$ induced by the closed, convex cones $(C(x))_{x \in \mathcal{X}}$, the following are equivalent.*

(i) *The relation is transitive.*

(ii) *If $y \succsim z$, then $C(y) \subseteq C(z)$.*

(iii) *If $y \succsim z$, then $C(z)^\circ \subseteq C(y)^\circ$.*

Proof. First, we show that (ii) implies (i). To this end, suppose $x \succsim y \succsim z$. Hence, it holds that $C(y) \subseteq C(z)$, $x \in y + C(y)$ and $y \in z + C(z)$. Therefore, we can conclude $x \in z + C(y) + C(z) \subseteq z + C(z) + C(z) = z + C(z)$ where the equality holds as $C(z)$ is a closed, convex cone and hence closed under addition.

For the implication of (i) to (ii), we use the characterization of a closed, convex cone being equal to its recession cone (compare [69, page 6 equation (1.3)]). Let $\succsim$ be transitive and suppose $y \succsim z$. Hence $y \in z + C(z)$, that is $y = z + w$ for some $w \in C(z)$. Fixing $v \in C(y)$, it holds that $tv \in C(y)$ for every $t > 0$ implying $tv + y \succsim y$. By transitivity, it follows that $tv + y \succsim z$ and hence $tv + y \in z + C(z)$ for every $t > 0$. Combining this with the former, we obtain $tv + z + w \in z + C(z)$ and thus $tv + w \in C(z)$ for every $t > 0$. Therefore, for v we can find $w \in C(z)$ such that $tv + w \in C(z)$ for every $t > 0$. By [69, page 6 equation (1.3)], it follows that v is in the recession cone of $C(z)$ and since $C(z)$ is a closed convex cone, it follows that $v \in C(z)$. This shows that $C(y) \subseteq C(z)$, as v was arbitrary.

The implication of (ii) to (iii) follows, as $C(y) \subseteq C(z)$ implies $C(z)^\circ \subseteq C(y)^\circ$ by applying the definition of the polar cone operator. Reversely, $C(z)^\circ \subseteq C(y)^\circ$ implies $C(y)^{\circ\circ} \subseteq C(z)^{\circ\circ}$. The Bipolar theorem states that a closed convex cone is equal to its bipolar, causing $C(y) = C(y)^{\circ\circ}$ and $C(z) = C(z)^{\circ\circ}$ which finishes the proof. □

Remark 5.5. The proof of the previous lemma was done independently and without knowledge of the proof in [36] although sharing the main idea.

Note that for checking the property of $y \succsim z$ implying $C(y) \subseteq C(z)$, for all $y, z \in \mathcal{X}$, it suffices to consider elements with $C(y) \neq \{0\}$ and $C(z) \neq \{0\}$. Indeed, if $C(y) = \{0\}$, then $C(y)$ is certainly a subset of $C(z)$ and if $C(z) = \{0\}$, then $y \succsim z$ reads as $y \in z + C(z) = \{z\}$ and thus $y = z$. Therefore, in the following examples we only consider the case of cones not being $\{0\}$ for the examined elements.

Example 5.6. In $\mathbb{R}^2$, let $C(0) = \mathbb{R}^2_+$, $C(x) = \{(v, w) \in \mathbb{R}^2_+ : w \geq \|x\|\, v\}$ for $x \in \mathbb{R}^2_+ \backslash \{0\}$ and $C(x) = \{0\}$ for $x \notin \mathbb{R}^2_+$. From now on, the considered elements are all in $\mathbb{R}^2_+$, since the cones are zero otherwise. First, we note that $\|x\| \geq \|y\|$ implies $C(x) \subseteq C(y)$. Second, $x \succsim y$, that is $x - y \in C(y)$, implies $x - y \in \mathbb{R}^2_+$ and thereby $\|x\| \geq \|y\|$. Therefore, $x \succsim y$ implies $C(x) \subseteq C(y)$, showing the transitivity of the relation.

The next example shows that Lemma 5.4 is not valid if the property of being a cone is dropped.

Example 5.7. Consider $\mathcal{X} = \mathbb{R}^2$. Let $C(1,1) = \{(x_1, x_2) : x_2 \geq 0\}$, $C(2,2) = \operatorname{conv}((0,0), (-1,-1))$ and the unspecified cones be zero which are closed, convex sets containing zero. We note that $C(2,2) \not\subseteq C(1,1)$ and $(2,2) + C(2,2) \subseteq (1,1) + C(1,1)$. Thus, we obtain a transitive relation such that $(2,2) \succsim (1,1)$ but not $C(2,2) \subseteq C(1,1)$.

The following two examples illustrate that dropping the property of closedness we cannot prove the equivalence of Lemma 5.4. Hence, the transitivity for a representation induced by moving cones which are not closed is difficult to characterize in terms of the cones only. Both examples have to be understood as relations in $\mathbb{R}^2$ induced by moving cones where all unspecified cones are zero. The first example shows that for a relation induced by moving cones, even if $y \succsim z$ always implies $C(y) \subseteq C(z)$, the relation does not have to be transitive.

Example 5.8. For $z = (1,2)$ define $C(z) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0 \text{ or } x_1 = 0 \text{ and } x_2 \geq 0\}$ and for $y = (1,3)$ let $C(y) = \mathrm{cl}(C(z))$. Then, it holds $y - z = (0,1)$ which is in $C(z)$ and hence $y \succsim z$. Considering $x = (1,1)$, it holds that $x - y = (0,-2) \in C(y)$ and $x - z = (0,-1) \notin C(z)$ meaning $x \succsim y$ but $x \not\succsim z$. Therefore, $\succsim$ cannot be transitive.

The next example illustrates that the case of $y \succsim z$ and $C(z) \subset C(y)$ may appear for transitive relations.

Example 5.9. Define $C((0,0)) = \mathrm{Int}\,(\mathbb{R}^2_+) \cup \{(0,0)\}$ and $C(x) = \mathbb{R}^2_+$ for $x \in \mathbb{R}^2_+ \setminus \{(0,0)\}$. It holds that $(1,1) - (0,0) \in C((0,0))$ meaning $(1,1) \succsim (0,0)$. However, it holds that $C((0,0)) \subset \mathrm{cl}(C(0,0)) = C((1,1))$. To show transitivity, consider $y \succsim z$ that is $y \in z + C(z)$. If $z \in \mathbb{R}^2_+ \setminus \{(0,0)\}$, then it follows that $y \in \mathbb{R}^2_+ \setminus \{(0,0)\}$ and thereby $C(y) = \mathbb{R}^2_+ = C(z)$. Hence, it holds that $y + C(y) \subseteq z + C(z)$. If $z = (0,0)$, then $y \in \mathrm{Int}\,(\mathbb{R}^2_+) \cup \{(0,0)\}$. In the case that $y \neq (0,0)$, it follows that $C(y) = \mathbb{R}^2_+$ and hence $y + C(y) \subseteq z + C(z)$. Therefore, Lemma 5.1 yields the transitivity property.

Definition 5.10. Let the relation $\succsim$ be induced by the collection of convex cones $(C(x))_{x \in \mathcal{X}}$, all containing zero. We call the relation $\succsim_{\mathrm{cl}}$ induced by the collection $(\mathrm{cl}(C(x))_{x \in \mathcal{X}}$ the continuous extension of $\succsim$.

As $x \succsim_{\mathrm{cl}} y$ if and only if $x - z \in \mathrm{cl}(C(y))$, $x \succsim y$ always implies $x \succsim_{\mathrm{cl}} y$ which indeed renders $\succsim_{\mathrm{cl}}$ to be an extension of $\succsim$.

Example 5.11. In $\mathbb{R}^2$, define $C(0,1) = \{(x,y) : y \geq x^-\}$, $C(0,0) = \mathrm{Int}\,(\mathbb{R}^2_+) \cup \{(0,0)\}$ and let the unspecified cones be equal to zero. This defines a transitive relation, since $(0,1)$ and $(0,0)$ cannot be compared and hence there do not exist chains $x \succsim y \succsim z$ of pairwise different elements which could contradict transitivity. Considering the continuous extension yields $(-1,2) \succsim_{\mathrm{cl}} (0,1) \succsim_{\mathrm{cl}} (0,0)$. However, $(-1,2) \not\succsim_{\mathrm{cl}} (0,0)$ contradicts the transitivity of $\succsim_{\mathrm{cl}}$. Hence, there exist transitive relations the continuous extension of which is nontransitive.

The Example 5.9 illustrated that there exist transitive continuous extensions of non-transitive relations. Thus, the transitivity of a relation and its extension are a priori unrelated. However, if the extension is transitive, we obtain the following result.

Lemma 5.12. *Let $\succsim$ be induced by convex cones containing zero. If the continuous extension $\succsim_{\mathrm{cl}}$ is transitive, then $y \succsim z$ implies $C(z)^\circ \subseteq C(y)^\circ$.*

Proof. Consider $y \succsim z$. Hence, it also holds $y \succsim_{\mathrm{cl}} z$. As $\succsim_{\mathrm{cl}}$ is transitive, the previous lemma implies that $(\mathrm{cl}(C(z)))^\circ \subseteq (\mathrm{cl}(C(y)))^\circ$. Since for every convex set C it holds that $C^\circ = (cl(C))^\circ$, the claim follows. $\square$

5.2.1 When is a Relation Induced by Moving Cones?

A relation is induced by moving cones if $C(y) = U(y) - y = \{x - y : x \succsim y\}$ is a cone. Hence, there is a representation induced by (closed, convex) moving cones if $U(y) - y$ is always a (closed, convex) cone. In terms of $\succsim$, we characterize this as follows.

Lemma 5.13. *A reflexive relation $\succsim$ is induced by moving cones if and only if for any y it holds that*

$$x \succsim y \quad \Longrightarrow \quad tx + (1-t)y \succsim y, \text{ for all } t > 0. \tag{5.2}$$

Moreover, the moving cones are convex if in addition one of the following equivalent assertions is fulfilled:

(i) All upper level sets are convex, meaning that for every y and $\lambda \in [0,1]$ it holds that

$$x_1, x_2 \succsim y \quad \Longrightarrow \quad \lambda x_1 + (1-\lambda)x_2 \succsim y.$$

(ii) For every $y \in \mathcal{X}$ it holds that

$$x_1, x_2 \succsim y \quad \Longrightarrow \quad x_1 + x_2 - y \succsim y.$$

Furthermore, all moving cones are closed if and only if all upper level sets are so.

Proof. First, we show that a relation is induced by moving cones if and only if (5.2) is fulfilled. Suppose the relation is induced by moving cones. Therefore, $U(y) - y$ is a cone and hence for every $t > 0$ and $w \in U(y) - y$, it follows $tw \in U(y) - y$. Considering $x \succsim y$, it follows that $x - y \in U(y) - y$ and hence $t(x - y) \in U(y) - y$, for every $t > 0$. Consequently, $t(x - y) + y \succsim y$ which yields (5.2). Reversely, suppose (5.2) is fulfilled. Consider $w \in U(y) - y$ that is $w + y \succsim y$. Hence, for $t > 0$ it holds that $tw + y \succsim y$, showing $tw \in U(y) - y$. Therefore, the relation is induced by moving cones.

It holds that $U(y) - y$ is convex if and only if $U(y)$ is so, showing the equivalence of convexity of the cones and Assertion (i). To prove the equivalence of convexity and Assertion (ii), we use the fact that a cone is convex if and only it is closed under addition. Suppose that the relation is induced by moving convex cones. This implies that $U(y)-y$ is closed under addition for every $y \in \mathcal{X}$. Hence, if $w_1, w_2 \in U(y) - y$, then it holds that $w_1 + w_2 \in U(y) - y$. Starting with $x_1, x_2 \succsim y$, it follows that $x_1 - y, x_2 - y \in U(y) - y$. Hence, we obtain that $x_1 + x_2 - 2y \in U(y) - y$, which translates to $x_1 + x_2 - y \succsim y$, showing Assertion (ii). Reversely, suppose that Assertion (ii) is fulfilled. Considering $w_1, w_2 \in U(y) - y$ yields that $w_1 + y, w_2 + y \succsim y$. Thus, by Assertion (ii) it follows that $w_1 + w_2 + y \succsim y$ and thereby $w_1 + w_2 \in U(y) - y$, showing the closedness under addition.

The remaining claim follows, since in a topological vector space a set $U(y)$ is closed if and only if $U(y) - y$ is so, as the addition operation is continuous. $\square$

Let a relation be induced by the closed convex cones $(C(x))_{x\in\mathcal{X}}$. For fixed $x \in \mathcal{X}$, consider the relation induced by $C(x)$ for which, due to Lemma 5.2, there exists an MU, denoted by U_x. Therefore, we obtain a representation of the form

$$x \succsim y \quad \Longleftrightarrow \quad u(x) \geq u(y), \ \forall u \in U_y. \tag{5.3}$$

Hence, a relation is induced by moving closed convex cones if and only if for every x there exists a set U_x of continuous, linear functions fulfilling (5.3).

5.2.2 Moving Cones and the Independence Axiom

A relation $\succsim$ on a vector space $\mathcal{X}$ fulfills the independence axiom if

$$p \succsim q \quad \Longrightarrow \quad \lambda p + (1-\lambda)r \succsim \lambda q + (1-\lambda)r, \ \forall r \in \mathcal{X}, \lambda \in [0,1].$$

We first examine the connection of a relation induced by moving convex cones and the independence axiom.

Lemma 5.14. *A reflexive relation is transitive and fulfills the independence axiom if and only if it is induced by a convex cone.*

Proof. We first show that the relation $\succsim$ induced by the convex cone C fulfills the independence axiom. To this end, let $p \succsim q$, that is $p \in q + c$, for $c \in C$ and consider an arbitrary $r \in \mathcal{X}$. It holds that $r \in r + C$ and since C is a convex cone, it follows that $\lambda p + (1-\lambda)r \in \lambda q + (1-\lambda)r + C$ and thereby $\lambda p + (1-\lambda)r \succsim \lambda q + (1-\lambda)r$. Moreover, a relation induced by the convex cone C is transitive (compare Lemma 5.20).

To prove the reverse implication, we first show that if $\succsim$ is reflexive, transitive and satisfies the independence axiom, then it is induced by moving convex cones $(C(p))_{p\in\mathcal{X}}$. We have to show that $U(q) - q$ is always a convex cone. As to the convexity, it is sufficient to show that $U(q)$ is convex. To this end, let $p_1 \succsim q$ and $p_2 \succsim q$. Applying the independence axiom twice yields that $\lambda p_1 + (1-\lambda)p_2 \succsim \lambda q + (1-\lambda)p_2 \succsim q$. The claim follows by transitivity. The positive homogeneity of $U(q) - q$ corresponds to $\lambda(U(q) - q) \subset U(q) - q$ or equivalently $\lambda U(q) + (1-\lambda)q \subseteq U(q)$. This however is a consequence independence axiom.

We finally show that if $\succsim$ is a reflexive, transitive relation induced by moving convex cones $(C(p))_{p\in\mathcal{X}}$ and satisfying the independence axiom, then all cones are equal. To this end, suppose that there exist $q, \tilde{q}$ such that $C(q) \neq C(\tilde{q})$. Pick r and $\lambda \in (0,1)$ such that $\tilde{q} = \lambda q + (1-\lambda)r$. Therewith, it holds that $C(q) \neq C(\lambda q + (1-\lambda)r)$ and without loss of generality there is some $y \in C(q) \setminus C(\lambda q + (1-\lambda)r)$. Defining $\tilde{y} = y + \lambda q + (1-\lambda)r$ yields that $\tilde{y}$ is in $\lambda q + (1-\lambda)r + C(q)$ but not in $\lambda q + (1-\lambda)r + C(\lambda q + (1-\lambda)r)$. For

$p := (\tilde{y} - (1-\lambda)r)/\lambda$, we show that $p \in q + C(q)$. Indeed, observe that

$$p - q = \frac{\tilde{y} - (1-\lambda)r}{\lambda} - q = \frac{y + \lambda q}{\lambda} - q = \frac{y}{\lambda}.$$

Since y is in $C(q)$, so is y/λ, implying $p - q \in C(q)$.

Thus, it holds that $p \succsim q$ but $\lambda p + (1-\lambda)r = \tilde{y} \notin \lambda q + (1-\lambda)r + C(\lambda q + (1-\lambda)r)$, that is $\lambda p + (1-\lambda)r \not\succsim \lambda q + (1-\lambda)r$. Therefore, the independence axiom is not fulfilled, in contradiction to our assumptions. □

Let now X be a compact metric spaces, $C_b(X)$ the set of all continuous real-valued functions on X, $\mathrm{ca}\,(X)$ the set of all finite Borel measures on X and $\mathcal{M}(X)$ the set of all Borel probability measures on X equipped with topology of weak convergence. A relation on $\mathcal{X}$ fulfills the continuity axiom if for every converging sequences $(p_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$ fulfilling $p_n \succsim q_n$, for every $n \in \mathbb{N}$, it holds that $\lim p_n \succsim \lim q_n$. In [34] the following result is proven.

Lemma 5.15. *A relation $\succsim$ on $\mathcal{M}(X)$ is reflexive, transitive and satisfies the independence and continuity axioms if and only if there is a closed, convex set $U \subset C_b(X)$ such that*

$$p \succsim q \quad \Longleftrightarrow \quad \int_X u dp \geq \int_X u dq, \ \forall u \in U.$$

Using the same arguments, it is possible to prove the following result without requiring transitivity.

Lemma 5.16. *Let $\succsim$ be a relation on $\mathrm{ca}\,(X)$ induced by moving convex and weak*-closed cones $(C(q))_{q\in \mathrm{ca}\,(X)}$. Then, there exist sets of functions $V_q \subseteq C_b(X)$, $q \in \mathrm{ca}\,(X)$, such that*

$$p \succsim q \quad \Longleftrightarrow \quad \int_X u dp \geq \int_X u dq, \ \forall u \in V_q. \tag{5.4}$$

Proof. Define $V_q = \{u \in C_b(X) : \int_X u d\mu \geq 0 : \forall \mu \in C(q)\}$. This set fulfills the desired property. □

Remark 5.17. We showed that a relation induced by moving convex cones has a representation in terms of (5.4). If the relation is transitive and fulfills the independence axiom, then all cones coincide and we recover the result of [34]. Moreover, any transitive relation fulfilling the independence axiom is induced by a moving convex cone.

In [34] it is shown that the continuity and independence axiom imply the weak*-closedness of a special cone, which is the main and delicate part to obtain a representation. The result is shown for relations on $\mathcal{M}(X)$ by extending the space to $\mathrm{ca}\,(X)$. The results of this section clarified the connection of the representation given in [34] and relations induced by moving cones.

5.2.3 Prospect: Nesting a Relation Induced by Moving Cones

The lexicographical order is induced by a convex cone, for instance in $\mathbb{R}^2$ by $C = \{(v,w) \in \mathbb{R}^2 : v > 0 \text{ or } v = 0 \text{ and } w \geq 0\}$. Unfortunately, C is not closed. Taking the closure, we obtain the cone $\mathrm{cl}(C) = \{(v,w) \in \mathbb{R}^2 : v \geq 0\}$. Moreover, C contains closed, convex cones, for example $C_n = \{(v,w) \in \mathbb{R}^2 : v \geq 0, w \geq -nv\}$, $n \in \mathbb{N}$. Therefore, it holds that $C_n \subseteq C \subseteq \mathrm{cl}(C)$, a property we describe by "C being nested by C_n and $\mathrm{cl}(C)$". The induced relations fulfill

$$x \succsim_{C_n} y \quad \Longrightarrow \quad x \succsim_C y \quad \Longrightarrow \quad x \succsim_{\mathrm{cl}(C)} y.$$

Setting $u_1(x) = x_1$ and $u_2(x) = x_2 + nx_1$, we obtain an MU for $\succsim_{C_n}$ and via $v_1(x) = x_1$ an MU for $\succsim_{\mathrm{cl}(C)}$. Therefore, the lexicographical order is nested by relations both of which allow for an MU.

In general, starting with some relation R, the aim is to find relations R_1 and R_2 with MUs such that $R_1 \subseteq R \subseteq R_2$. If $(u_i)_{i \in I}$ denote the utility functions for R_1 and $(v_j)_{j \in J}$ for R_2, it holds that

$$u_i(x) \geq u_i(y),\ \forall i \in I \quad \Longrightarrow \quad xRy \quad \Longrightarrow \quad v_j(x) \geq v_j(y),\ \forall j \in J.$$

Example 5.18. Let a relation $\succsim$ be induced by the moving convex cones $(C(x))_{x \in \mathcal{X}}$. Suppose that $C_2 := \cup_{x \in \mathcal{X}} C(x) \neq \mathcal{X}$ is a closed, convex cone and $C_1 := \cap_{x \in \mathcal{X}} C(x) \neq \{0\}$. It holds that $C_1 \subseteq C(x) \subseteq C_2$ for every $x \in \mathcal{X}$. Hence, defining $x \succsim_j y$ if and only if $x - y \in C_j$, $j = 1, 2$ it holds that

$$x \succsim_1 y \quad \Longrightarrow \quad x \succsim y \quad \Longrightarrow \quad x \succsim_2 y,$$

and hence $\succsim$ is nested by two relations with MUs. As an example, let in $\mathbb{R}^2_+$ be $C(x) = \{(v,w) \in \mathbb{R}^2 : v \geq 0, w \geq \min(v/\|x\|, v/2)\}$ for every x in $\mathbb{R}^2_{++}$ and let the remaining cones be $\mathbb{R}^2_+$. Then, $C_1 = \{(v,w) \in \mathbb{R}^2 : v \geq 0, w \geq v/2)\}$ and $C_2 = \mathbb{R}^2_+$.

Definition 5.19. Let $\succsim$ be a relation on $\mathcal{X}$ and $A \subseteq \mathcal{X}$ a nonempty subset. An element $y \in \mathcal{X}$ is said to be nondominated on A if there is no $x \in A$ such that $x \succ y$ and $x \in \mathcal{X}$ is said to be nondominating on A if $x \not\succ y$ for all $y \in A$.

From an optimization perspective the situation of a relation being nested by two others has the following implication. Suppose that $R_1 \subseteq R \subseteq R_2$ and R_1 and R_2 are antisymmetric. Then every nondominated set with respect to R_2 is also nondominated with respect to R. Thus, for maximization purposes we can conclude R from R_2. Reversely, a nondominating element with respect to R_1 is also nondominating with respect to R. Hence, for minimization purposes we can conclude R from R_1. In the case of the relations R_1 and R_2 being induced by the closed, convex cones C_1 and C_2, respectively, the antisymmetry property corresponds to $C_1 \cap (-C_1) = C_2 \cap (-C_2) = \{0\}$.

5.3 Moving Convex Sets and Transitivity

Throughout this section, $\mathcal{X}$ denotes a locally convex topological vector space. We first examine relations induced by a convex set $C \subseteq \mathcal{X}$ containing zero, that is $x \succsim y$ if and only if $x - y \in C$.

Lemma 5.20. *The relation induced by C is transitive if and only if C is a cone.*

Proof. Since a cone is convex if and only if it is closed under addition, it is sufficient to show that $\succsim$ is transitive if and only if $C + C \subseteq C$. Suppose that $C + C \subseteq C$ and $x \succsim y \succsim z$, that is $x - y, y - z \in C$. Therefore, $x - z = (x - y) + (y - z)$ is in $C + C$ and hence in C. This yields $x \succsim z$, showing the relation to be transitive. Now suppose that the relation is transitive and let $c_1, c_2 \in C$. Thus, $c_1 - 0, 0 - (-c_2) \in C$, that is $c_1 \succsim 0 \succsim -c_2$. Due to transitivity, it follows that $c_1 \succsim -c_2$ yielding $c_1 + c_2 \in C$. □

Let A be a subset of the convex set C such that $A + A \subseteq C$. Then $x - y \in A$ and $y - z \in A$ always implies $x - z \in C$. Hence, $x \succsim y \succsim z$ implies $x \succsim z$, provided $x - y \in A$ and $y - z \in A$. This is an example of a relation which is not transitive everywhere however on special subsets it is. The set $A = 0.5C$ fulfills $A + A \subseteq C$, since C is convex and contains zero, and $0.5C$ is the largest possible set fulfilling this.

Note that if $C = \mathcal{X}$, we have $x \succsim y$ for every $x, y \in \mathcal{X}$. Moreover, C is closed, convex and $C \neq \mathcal{X}$ if and only if it is the intersection of all closed half spaces containing it (compare [3, Corollary 5.83]). Consequently,

$$C = \bigcap_{x^* \in \mathcal{X}^*} H(x^*),$$

with half spaces

$$H(x^*) = \{x \in \mathcal{X} : x^*(x) \geq \alpha(x^*)\},$$

for the dual elements $x^* \in \mathcal{X}^*$ and $\alpha(x^*) = \inf_{x \in C} x^*(x)$, $x^* \in \mathcal{X}^*$. Therefore,

$$x \succsim y \iff x^*(x) \geq x^*(y) + \alpha(x^*), \ \forall x^* \in \mathcal{X}^*. \tag{5.5}$$

The above describes how to obtain a dual representation (5.5) for relations induced by closed, convex sets. For a particular chain $x \succsim y \succsim z$ with $x^*(x) \geq x^*(y) + 0.5\alpha(x^*)$ and $x^*(y) \geq x^*(z) + 0.5\alpha(x^*)$, for all $x^* \in \mathcal{X}^*$, it follows $x \succsim z$. This corresponds to the transitive subset induced by $0.5C$, since $0.5C$ is the intersection of all $\{x \in \mathcal{X} : x^*(x) \geq 0.5\alpha(x^*)\}$, $x^* \in \mathcal{X}^*$.

5.3.1 Transitivity of Relations Induced by Moving Convex Sets

In this subsection, we study the transitivity property for the special class of relations induced by the moving convex sets

$$C(x) = t(x)C + m(x),\ x \in \mathcal{X}$$

where C is a fixed closed convex subset of a topological vector space $\mathcal{X}$ containing zero and $m : \mathcal{X} \to \mathcal{X}$ and $t : \mathcal{X} \to \mathbb{R}_+$ are two functions. Thus, C is shrunk or enlarged by the factor $t(x)$ and then shifted by $m(x)$. The relation is reflexive if and only if $C(x)$ contains zero, that is $m(x) \in -t(x)C$ for every $x \in \mathcal{X}$ and we only consider this case. Moreover, the relation has closed convex upper level sets, since $U(x) = x+m(x)+t(x)C$, $x \in \mathcal{X}$. We attend to the question of when a relation induced by these moving closed convex sets is transitive.

Remark 5.21. A relation which both provides an MU representation by $(u_i)_{i\in I}$ and is induced by moving convex sets $(C(x))_{x\in\mathcal{X}}$ fulfills

$$x \succsim y \iff x \in y + C(y) \iff u_i(x) \geq u_i(y), \forall i \in I.$$

If a relation has an MU of quasiconcave utility functions, it is transitive and induced by moving convex sets, since $C(x) = \{x : u_i(x) \geq u_i(y), \forall i \in I\} - y$ is convex in that case. Reversely, every transitive relation induced by moving convex sets has convex upper level sets and thus can be represented by the indicator functions of the upper level sets which is a quasiconcave MU. Hence, showing a relation induced by moving convex sets to be transitive is equivalent to showing it to have a quasiconcave MU. In the previous section, we proved that there exists a description of transitivity if every $C(x)$, $x \in \mathcal{X}$, is a cone. Dropping the property of being a cone this description cannot be applied anymore and to give sufficient conditions for transitivity of relations induced by moving convex sets becomes delicate. Therefore, we examine a specific case.

Let us illustrate the specific type of relation induced by $C(x) = t(x)C + m(x)$ by a few examples.

Example 5.22. Denote $[x]_{up} = \min\{n \in \mathbb{N} : n > x\}$ and $[x]_{down} = \max\{n \in \mathbb{N} : n \leq x\}$ for $x \in \mathbb{R}$. The functions $x \mapsto [x]_{down}$ and $x \mapsto -[x]_{up}$ are quasiconcave. The relation corresponding to the MU of these functions is given by $y \succsim x$ if both $[y]_{down} \geq [x]_{down}$ and $[x]_{up} \geq [y]_{up}$. Therefore, the relation is also given by the moving convex sets $C(x) = [[x]_{down} - x, [x]_{up} - x]$ and $\mathbb{R}$ is divided in equivalence classes of the form $[[x]_{down}, [x]_{up}]$, $x \in \mathbb{R}$. On $\mathbb{R}^2$, the relation represented by the four quasiconcave utility functions mapping (x_1, x_2) to $-[x_1]_{up}, [x_1]_{down}, -[x_2]_{up}$ or $[x_2]_{down}$, respectively, is also induced by $C(x) = \{(v, w) : [x_1]_{down} - x_1 \leq v \leq [x_1]_{up} - x_1, [x_2]_{down} - x_2 \leq [x_2]_{up} - x_2\}$, which are squares of length 1. Note that $C(x) = C+m(x)$ for $C = [0, 1]$ and $m(x) = [x]_{down}-x$

in the one-dimensional case and $C = [0,1]^2$ and $m(x) = ([x_1]_{down} - x_1, [x_2]_{down} - x_2)$ in the two-dimensional one.

In $\mathbb{R}$ consider $C(x) = t(x)C$ for $t(x) = |x|$ and $C = [0,1]$ or equivalently $C(x) = [0,|x|]$ and thereby $U(x) = [x, x + |x|]$. Then, there exists an MU by $u_1(x) = x$ and $u_2(x) = -x - |x|$. Indeed if $u_1(x) \geq u_1(y)$, it holds that $x \geq y$. If $u_2(x) \geq u_2(y)$, then it holds that $-x - |x| \geq -y - |y|$, thereby $x + |x| \leq y + |y|$ and thus $x \leq y + |y|$.

As the final example, let $u_1(x) = -|x|$ and $u_2(x) = x$ which are quasiconcave functions. Then the relation corresponding to the MU of u_1 and u_2 has the upper level sets $U(y) = \{y\}$ if $y \geq 0$ and $U(y) = [y, -y]$ if $y < 0$. Hence, $C(y) = \{-y\}$ for $y \geq 0$ and $C(y) = [0, -2y]$ for $y < 0$ which can also be written as $C(y) = t(y)C + m_y$, with $C = [0,1]$, $t(y) = 0, m(y) = -y$ for $y \geq 0$ and $t(y) = -2y, m(y) = 0$ for $y \leq 0$.

We begin by considering a normed space where the fixed convex set is the closed unit ball.

Lemma 5.23. *In the normed space $(\mathcal{X}, \|.\|)$, let $B_r(x) = \{y \in \mathcal{X} : \|x - y\| \leq r\}$ and $C(x) = m(x) + t(x)B_1(0) = B_{t(x)}(m(x))$, $x \in \mathcal{X}$. Then, for the relation $\succsim$ induced by the moving convex sets $(C(x))_{x\in\mathcal{X}}$ the following are equivalent:*

(i) *The relation is transitive.*

(ii) *It holds that $x \succsim y$ if and only if $t(x) \leq t(y) - \|x + m(x) - (y + m(y))\|$.*

Proof. Due to the one-to-one relation of transitivity and upper level sets it suffices to show that $U(x) \subseteq U(y)$ if and only if $t(x) \leq t(y) - \|x + m(x) - y - m(y)\|$. By definition, it holds that $U(x) = x + B_{t(x)}(m(x)) = \{y \in \mathcal{X} : \|y - x - m(x)\| \leq t(x)\}$.

First, suppose $t(x) \leq t(y) - \|x + m(x) - y - m(y)\|$. Pick some $z \in U(x)$, thereby $\|z - x - m(x)\| \leq t(x)$, and hence

$$\|z - x - m(x)\| \leq t(x) \leq t(y) - \|x + m(x) - y - m(y)\|.$$

Applying the triangle inequality yields

$$\|z - y - m(y)\| \leq \|z - x - m(x)\| + \|x + m(x) - y - m(y)\| \leq t(y),$$

which shows that $z \in U(y)$.

Second, suppose that $t(x) > t(y) - \|x + m(x) - y - m(y)\|$. Pick z such that both $\|z - x - m(x)\| = t(x)$ and $x + m(x) = \lambda_0(y + m(y)) + (1 - \lambda_0)z$ are fulfilled for some $\lambda_0 \in [0,1]$ the existence of which follows by the projection on a closed convex set being continuous. Then it holds that

$$\begin{aligned}\|z-y-m(y)\| &= \lambda_0 \|z-y-m(y)\| + (1-\lambda_0)\|z-y-m(y)\| \\ &= \|\lambda_0(z-y-m(y))\| + \|(1-\lambda_0)(z-y-m(y))\| \\ &= \|z-x-m(x)\| + \|x+m(x)-y-m(y)\| \\ &= t(x) + \|x+m(x)-y-m(y)\| \\ &> t(x)+t(y)-t(x) \\ &= t(y).\end{aligned}$$

Hence, $\|z-y-m(y)\| > t(y)$, showing $z \notin U(y)$. However, since $\|z-x-m(x)\| = t(x)$, it holds that $z \in U(x)$. Thus, $U(x) \not\subseteq U(y)$ which yields the claim. □

Example 5.24. If $t(x) = 1$ for every x then the transitivity of the relation induced by the moving sets $C(x) = C + m(x)$ can be expressed by $x \succsim y$ if and only if $\|x+m(x)-y-m(y)\| = 0$ and hence $x+m(x) = y+m(y)$. Therefore, the space $\mathcal{X}$ is divided in equivalence classes.

If $m(x) = 0$ for all x, then the transitivity of the relation induced by the moving sets $C(x) = t(x)C$ can be expressed by $x \succsim y$ if and only if $\|x-y\| \le t(y)-t(x)$.

Lemma 5.23 dealt with C being a bounded set, a specific property playing a central role in characterizing transitivity.

Definition 5.25. Denote $\mathrm{Dir}\,(C) = \{x \in C : x + C \subseteq C\}$.

For $x \in \mathrm{Dir}\,(C)$ it holds that $tx \in C$, for all $t \ge 0$. Hence, $\mathrm{Dir}\,(C) = C$ if and only if C is a cone, as C is convex, and $\mathrm{Dir}\,(C) = 0$ if and only if C is bounded.

Lemma 5.26. *For a relation induced by the moving convex sets $C(x) = C + m(x)$, $x \in \mathcal{X}$, the following are equivalent:*

(i) The relation is transitive.

(ii) It holds that $x \succsim y$ if and only if $x + m(x) - y - m(y) \in \mathrm{Dir}\,(C)$.

If C is bounded, (ii) reads as $x \succsim y$ if and only if $x + m(x) = y + m(y)$.

Proof. We use that transitivity can be expressed by $x \succsim y$ if and only if $U(x) \subseteq U(y)$. The fact $U(x) \subseteq U(y)$ expresses that $x + m(x) + C \subseteq y + m(y) + C$. This is equivalent to $x + m(x) - y - m(y) + C \subseteq C$, implying $x + m(x) - y - m(y) \in \mathrm{Dir}\,(C)$. □

Remark 5.27. Suppose a set $\mathcal{X}$ is divided in equivalence classes $(M_i)_{i\in I}$, where I has the same cardinality as $\mathbb{R}$. Consider a function $f : I \to \mathbb{R}$ with $f(i) = f(j)$ if and only if $i = j$. Defining $u_1(x) = f(i)$ and $u_2(x) = -f(i)$, for $x \in M_i$, is an MU for this relation. Recall that a polytope is a set which is the convex hull of finitely many elements. In the previous lemma, it was shown that a convex set C which is not a polytope but bounded may induce an equivalence relation. Hence, the property of the convex set C being a polytope is not necessary to obtain a finite MU for the induced relation.

For the case $C(x) = t(x)C$, we first show a necessary condition.

Lemma 5.28. *Let $C(x) = t(x)C$ with $t(x) \geq 0$ for all $x \in \mathcal{X}$. Then, $U(x) \subseteq U(y)$ implies that either $t(x) < t(y)$ or $x - y \in \operatorname{Dir}(t(x)C)$. If C is bounded, then $x - y \in \operatorname{Dir}(t(x)C)$ is equivalent to $x = y$.*

Proof. The fact $U(x) \subseteq U(y)$ expresses $x - y + t(x)C \subseteq t(y)C$. Supposing $t(x) \geq t(y)$ yields $t(y)C \subseteq t(x)C$. Hence, $x - y + t(x)C \subseteq t(x)C$, in particular $x - y \in t(x)C$ and therefore $r(x - y) \in t(x)C$ for all $r \geq 0$. Thus, either $t(x) = 0$, that is $x = y$, or $x - y \in \operatorname{Dir}(t(x)C)$. □

Example 5.29. On $\mathbb{R}$ let $C = [1, \infty)$. Consider $x = 1, y = 0, t(x) = 1$ and $t(y) = 0.5$. Then it holds $0.5C = [0.5, \infty) \subseteq C$ and hence $x + t(x)C = 1 + C \subseteq 0.5C = y + t(y)C$. This illustrates how for a representation induced by $C(x) = t(x)C$ where C is not bounded, also the case $t(y) > t(x)$ has to be considered.

Definition 5.30. For a relation induced by $C(x) = t(x)C$ define

$$t_{\sup}(x, y) = \sup\{s \in \mathbb{R}_+ : C \subseteq s(y - x + t(y)C)\}.$$

The next lemma shows a condition for transitivity. However, specifying this condition was possible for normed spaces but is delicate in a general case. Indeed, for a given closed, convex set C and $z \in C$, the maximal s such that $z + sC \subseteq C$ needs to be determined.

Lemma 5.31. *Let a relation be given by moving convex sets of the form $C(x) = t(x)C$. If the relation is transitive and $t(x) > 0$, then it holds that $x \succsim y$ if and only if $(1/t(x)) \leq t_{\sup}(x, y)$.*

If C is bounded, then $x \succsim y$ implies $t(x) \leq t(y)$, provided the relation is transitive.

Proof. If $t(x) > 0$, it holds that $U(x) = x + t(x)C \subseteq y + t(y)C = U(y)$ if and only if $C \subseteq 1/t(x)(y - x + t(y)C)$. This can only be the case if $(1/t(x)) \leq t_{\sup}(x, y)$, showing the first claim.

To prove the second claim, we use that transitivity is equivalent to $x \succsim y$ if and only if $x + t(x)C \subseteq y + t(y)C$. In this situation, it holds $x = y + t(y)c$ with $c \in C$ and $y + t(y)c + t(x)C \subseteq y + t(y)C$ implying that $t(y)c + t(x)C \subseteq t(y)C$. If $t(x) = 0$, the claim is true and if $t(y) = 0$ it follows $x = y$ and thereby $t(x) = t(y)$. Suppose that neither $t(x)$ nor $t(y)$ are zero. Then we can divide by $t(y)$ and obtain $c + t(x)/t(y)C \subseteq C$. Since $c \in C$, it follows that $(1 + t(x)/t(y))c \in C$, hence also $(t(x)/t(y) + t(x)^2/t(y)^2)c \in C$ and by induction $[(t(x)/t(y))^n + (t(x)/t(y))^{n+1}]c \in C$ for every $n \in \mathbb{N}$. As C is bounded, this is only possible if $|t(x)/t(y)| \leq 1$ and hence $t(x) \leq t(y)$. □

5.4 Restricted Transitivity

A relation $\succsim$ is transitive if $x \succsim y \succsim z$ always implies $x \succsim z$. Given a chain $x_1 \succsim x_2 \cdots \succsim x_N$, $N \in \mathbb{N}$, of finitely many elements, it follows by induction that $x_1 \succsim x_N$. We use the term that a chain $x_1 \succsim \cdots \succsim x_N$ is transitive if it holds that $x_1 \succsim x_N$. For a relation $\succsim$ on $\mathcal{X}$ with corresponding $R = \{(x, y) : x \succsim y\}$ the transitive hull is given by

$$T(R) = \{(x, y) \in \mathcal{X} \times \mathcal{X} : \exists x_1, \dots, x_N, x \succsim x_1 \succsim \cdots \succsim x_N \succsim y\}.$$

The transitive hull is the smallest transitive relation containing $\succsim$. A relation is non-transitive if it does not coincide with its transitive hull or equivalently there exist chains which are not transitive.

Example 5.32. Let $R = \bigcup_{x \in \mathbb{R}}(x, [x-1, x])$, that is x is better than y for any $y \in [x-1, x]$. This relation is convex but not transitive. For $x^* \in \mathbb{R}$, let $\alpha(x^*) = x^*$ for $x^* \geq 0$ and $\alpha(x^*) = 0$ for $x^* < 0$. Then, xRy if and only if $x^*(x) - x^*(y) \leq \alpha(x^*)$ for any x^*. Hence, a dual representation is also possible for nontransitive relations.

There are basically two types of nontransitive relations. First, relations for which there appear rings $x \succsim y \succsim z \succ x$ and second, relations for which the first and the last element of a chain cannot be compared, meaning $x \succsim y \succsim z$ but $x \not\succsim z$ and $z \not\succsim x$. We attend to the second type throughout this section which may in particular occur for long chains $x_1 \succsim x_2 \succsim \cdots \succsim x_{1000}$ where it is impossible to compare x_1 and x_{1000}. We consider relations not all chains of which are transitive, but only certain ones for which the first and the last element are "similar". This "similarity" may stem from an a priori classification of the elements one compares. As an example, consider the ordering of groceries. A possible chain of comparison could be

$$coke \succsim soda \succsim milk \succsim cheese \succsim mozzarella \succsim salad \succsim apples.$$

Since coke, soda and milk are sorts of drinks, they are "similar" in this regard and hence one may compare them and prefers coke over milk. The same holds true for the chain of mozzarella, salad and apple, since they are different kind of food and also for the chain of milk, cheese and mozzarella, since they share the main ingredient. We obtain

$$coke \succsim milk \succsim mozzarella \succsim apples.$$

However, it seems to be difficult to compare groceries of different groups such as coke and apples, since they are rather "dissimilar". Hence, the obtained chain cannot be simplified as coke and mozzarella and milk and apples are "dissimilar" for the same reasons. Suppose in this situation, we compare apples to apple juice and obtain

$$coke \succsim milk \succsim mozzarella \succsim apples \succsim applejuice.$$

The situation changed, as now the first and the last element of the chain are both drinks, thus “similar” and comparable. Hence, we obtain *coke* $\succsim$ *applejuice*. However, this chain omits the comparison of for instance milk and apples. We exemplified the idea that we have an a priori understanding of which elements are “similar”, making it easier to compare them even if a long chain of comparison is in between both. Note that in this situation, there can appear chains only some elements of which can be compared but not all. The objective of this section is to formalize this feature.

5.4.1 The Feature s-Transitivity

Definition 5.33. Let $s : \mathcal{X} \times \mathcal{X} \to \{0, 1\}$ be a symmetric function such that $s(x, x) = 1$ for all $x \in \mathcal{X}$. A relation is s-transitive if from $x \succsim y_1 \succsim \cdots \succsim y_N \succsim z$, $N \in \mathbb{N}$, and $s(x, z) = 1$ it follows that $x \succsim z$. We will use the term that x and y are similar if $s(x, y) = 1$ and dissimilar if $s(x, y) = 0$.

Example 5.34. In $\mathbb{N}$, let $s(m, n) = 1$ if and only if $|m - n| \in 2\mathbb{N}$ that is both are odd or both even. Consider the relation $\succsim$ defined by $m \succsim n$ if and only if $m - n \in 2\mathbb{N} \cup \{0, 1\}$. Hence m is equivalent to itself, better than its direct predecessor and better then every even or odd predecessor of m if m is even or odd, respectively. This is an s-transitive relation, it is not transitive though, since for instance $4 \succsim 3 \succsim 1$ but $4 \not\succsim 1$. However, there are transitive chains such as $4 \succsim 3 \succsim 2$, since $4 \succsim 2$. Note that $4 \succsim 3 \succsim 3$ and $4 \succsim 3$ although $s(4, 3) = 0$.

The previous example illustrates that even if $s(x, z) = 0$, it is possible that $x \succsim y \succsim z$ and $x \succsim z$. Hence, dissimilarity of x and z need not to imply incomparability, that is $x \not\succsim z$ and $z \not\succsim x$. Reversely, similarity of x and y should not imply comparability. For example, the relation $x \succsim y$ if and only if $x = y$ is s-transitive for any s.

Remark 5.35. Note that any relation is s_1-transitive for the trivial function $s_1(x, y) = 1$ if and only if $x = y$. Moreover, for a relation $\succsim$ and corresponding $R = \{(x, y) : x \succsim y\}$ define the function s_2 by $s_2(x, z) = s_2(z, x) = 0$ if (x, z) or (z, x) is in $T(R) \setminus R$ and 1 otherwise. The relation is s_2-transitive and in the set of all functions s with respect to which the relation is s-transitive the set $A(s) = \{(x, y) : s(x, y) = 1\}$ is maximized for $s = s_2$.

The previous remark shows that given a relation, searching for functions s with respect to which it is s-transitive is only of minor interest. In the remainder of this section, we follow the converse idea. Given a function s, we examine relations which are s-transitive. A function s may be interpret as a tool to declare which elements fulfill the transitivity because they are sufficiently similar.

Remark 5.36. Let $\succsim$ be a reflexive relation on $\mathcal{X}$ which is s-transitive and $A \subseteq \mathcal{X}$ be a set such that $s(x, y) = 1$ for any $x, y \in A$. Denoting $R = \{(x, y) : x \succsim y\}$, the relation $\succsim|_{A \times A}$ is given by $R_A = R \cap (A \times A)$. In this case, the relation $\succsim|_{A \times A}$ is transitive.

Indeed, consider $x \succsim y \succsim z$ where $x, y, z \in A$. Since $s(x, z) = 1$, it follows $x \succsim z$ by s-transitivity.

5.4.2 Extensions of s-Transitive Relations

Throughout this subsection, let an arbitrary function s as in Definition 5.33 be given. Analogously to the transitive hull $T(R)$ of a relation R, we define the following.

Definition 5.37. Denote

$$ST(R) = \{(x, y) : (x, y) \in R, \text{ or } (x, y) \in T(R) \text{ and } s(x, y) = 1\}.$$

It can easily be verified that

$$ST(R) = R \cup \left(T(R) \cap s^{-1}(1)\right),$$

and this $ST(R)$ is the smallest s-transitive relation containing R. Moreover, it holds that $R \subseteq ST(R) \subseteq T(R)$.

Example 5.38. On $\mathbb{R}$, let $s(x, y) = 1$ if and only if $|x - y| \leq 2$. Define the relation $R = \cup_{x \in \mathbb{R}}(x, [x - 1, x])$. The transitive hull makes x to be preferred to $(-\infty, x]$. The hull $ST(R)$, however, makes x only to be preferred to $[x - 2, x]$.

Lemma 5.39. *A relation R is s-transitive if and only if $s(T(R) \setminus R) = 0$.*

Proof. Let R be s-transitive. Then, R coincides with $ST(R) = R \cup (T(R) \cap s^{-1}(1))$ and thus $(T(R) \setminus R) \cap s^{-1}(1) = \emptyset$. As s can only take two values, the claim follows. Reversely, suppose $s(T(R) \setminus R) = 0$. Hence, there does not exist a chain $xRx_1R \ldots x_NRy$ with $(x, y) \notin R$ and $s(x, y) = 1$. Therefore, it is impossible to find a chain contradicting s-transitivity, showing the claim follows. □

Lemma 5.40. *The intersection of s-transitive relations is s-transitive. Moreover, the union of a chain of s-transitive relations is s-transitive.*

Proof. The first claim follows by the characterization of s-transitivity in the previous lemma. Indeed, $s(T(\cap R_i) \setminus \cap R_i) = s(\cap T(R_i) \setminus R_i) = 0$ for arbitrary s-transitive relations R_i. For the second claim, let a chain (R_i) of s-transitive relations be given. We need to show that $x(\cup R_i)y(\cup R_i)z$ and $s(x, z) = 1$ implies $x(\cup R_i)z$. The former implies the existence of R_j and R_k such that xR_jyR_kz. Since we consider a chain of sets, it follows that $R_j \subseteq R_k$ or vice versa. In both cases, the claim follows by using s-transitivity of R_j or R_k, respectively. □

Proposition 5.41. *A reflexive, s-transitive relation R is the intersection of all complete, s-transitive relations containing it.*

Proof. Fix $(x, y) \notin R$ and consider

$$\left\{\tilde{R} : R \subseteq \tilde{R}, \tilde{R} \text{ is } s\text{-transitive}, (x, y) \notin \tilde{R}\right\}.$$

As all properties are preserved for union of chains, by Zorn's lemma there exists a maximal relation Q which is s-transitive, $R \subseteq Q$ and Q does not contain (x, y). We claim that Q is complete. Suppose to the contrary that there exist $(a, b), (b, a) \notin Q$. Defining $ST_1 = ST(Q \cup (a, b))$ and $ST_2 = ST(Q \cup (b, a))$ gives $ST_1 \cap ST_2 = Q$. Indeed, let (v, w) be in the intersection but not in Q. Since Q is transitive, the cases $(v, w) = (a, b)$ or $(v, w) = (b, a)$ can only appear if $s(a, b) = 1$. Hence, $s(v, w) = s(w, v) = 1$ and there exist sequences (x_i) and (y_i) with $v(ST_1)x_1(ST_1) \ldots x_n(ST_1)w$ and $v(ST_2)y_1(ST_2) \ldots y_m(ST_2)w$, respectively. If each pair of one of the chains is in fact in Q, it follows that $vQ \ldots Qw$ and since $s(v, w) = s(w, v) = 1$ and Q is s-transitive, this leads to $(v, w) \in Q$, a contradiction. Hence, there has to appear (a, b) or (b, a), respectively, in the chains. Therefore, it holds $vQx_1 \ldots x_kQa$ and $aQy_l \ldots y_mQw$ for some k and l. Hence, we have $vQ \ldots Qw$ and $s(v, w) = 1$, implying $(v, w) \in Q$ as Q is s-transitive. Consequently, either ST_1 or ST_2 does not contain (x, y), contradicting the maximality of Q. Thus, Q has to be complete. Hence, starting with $(x, y) \notin R$ we find a complete, s-transitive relation $Q(x, y)$ containing R but not (x, y). This finally shows that $R = \cap\{Q(x, y) : (x, y) \notin R\}$ and finishes the proof. $\square$

Note that there might exist complete, s-transitive extensions of R being transitive. Moreover, for a pair $(s, t) \in T(R) \setminus R$ there has to exist an extension fulfilling $s \succsim_1 y \succsim_1 t, s \succsim_1 t$ and another extension with $s \succsim_2 y \succsim_2 t \succ_2 s$.

5.4.3 Prospect: Representations of s-Transitive Relations

To derive a representation for s-transitive relations one has to handle two important issues. First, an s-transitive relations may or may not be transitive. Hence, a representation for s-transitive relations has to be applicable to transitive relations without enforcing transitivity. Second, the question whether x and y can be compared by $\succsim$ should be independent of whether $s(x, y)$ is one or zero.

We examine whether a modified threshold representation can be used. Consider

$$x \succ y \quad \Longleftrightarrow \quad u(x) > u(y) + c - s(x, y), \tag{5.6}$$

with $c > 1$ (due to reflexivity). This makes it impossible to distinguish the incomparable elements from the equivalent ones, but neither $s(x, y) = 0$ nor $s(x, y) = 1$ directly implies that x and y can or cannot be compared. A relation with such a representation fulfills s-transitivity. Indeed, $x \succ y \succ z$ means $u(x) > u(y) + c - s(x, y)$ and $u(y) > u(z) + c - s(y, z)$. Hence, $u(x) > u(z) + 2c - s(x, y) - s(y, z) > u(z) + 2c - 2$. In the case that $s(x,z) = 1$, the claim follows if $2c - 2 > c - 1$, which is equal to $c > 1$. Note that if $c > 2$ we also have transitivity for $s(x,z) = 0$. Hence, the representation can be applied to transitive relations. Nevertheless, in the best of all cases, namely a complete and transitive relation, the normal utility cannot be used as a representation. Indeed, although it holds that $x \succ y$ if $u(x) > u(y)$ the property (5.6) is only fulfilled if $\inf_{x \succ y}[u(x) - u(y)] > 0$, which in general is not the case. Another form of representation is given by

$$x \succ y \quad \Longleftrightarrow \quad u(x) > u(y) + b(c - s(x,y)), \tag{5.7}$$

with $c > 1, b \geq 0$. This works, since b can be zero and hence the normal utility can be used. Yet another possibility is close to the original threshold representation:

$$x \succ y \quad \Longleftrightarrow \quad u(x) > u(y) + t(y)(c - s(x,y)), \tag{5.8}$$

with $c > 1$ and $t(y) \geq 0$. The representations (5.6), (5.7) and (5.8) share the main idea of s-transitivity: $s(x,y) = 1$ should not expresses that one can compare x and y but rather it should make a comparison "easier". For instance in (5.6), this "easier" appears as $u(x) - u(y)$ has to be greater than c if $s(x,y) = 0$ but only greater than $c - 1$ if $s(x,y) = 1$. Both cases can appear and so this property can be used to verify the usefulness of a representation.

Finally, we examine whether we can use a set-valued representation. We know that for reflexive, transitive relations it holds $x \succsim y$ if and only if $U(x) \subseteq U(y)$. In particular, $x \succsim y$ implies $U(x) \subseteq U(y)$ and $x \succ y$ implies $U(x) \subset U(y)$. Can we find an appropriate characterization for s-transitive relations? Defining $S(y) = \cap\{z \in \mathcal{X} : s(z,y) = 1\}$, it holds that

$$x \succsim y \quad \Longrightarrow \quad U(x) \cap S(y) \subseteq U(y), \tag{5.9}$$

$$x \succ y \quad \Longrightarrow \quad U(x) \cap S(y) \subset U(y). \tag{5.10}$$

An intersection with $S(y)$ on the right-hand side of (5.9) and (5.10) does not change the implication, as $y \in S(y)$. Due to s-transitivity, (5.9) follows immediately. To obtain (5.10), we use that $x \succ y$ implies that y is not in $U(x)$ but in $U(y)$. Since the left-hand side also depends on y and not only on x, this characterization is not very meaningful. The reverse implications in both (5.9) and (5.10) can only be shown for elements x with $s(x,y) = 1$, since in that case $x \in U(x) \cap S(y)$. Hence, we only obtain a one-to-one relation for elements with $s(x,y) = 1$. Since y is in $S(y)$, either $y \in U(x) \cap S(y)$, that is $y \succsim x$, or $U(x) \cap S(y) \subset U(x)$. Supposing $U(x)$ and $S(x)$ to be convex and closed, one could separate $U(x) \cap S(y)$ and $U(y)$ by a functional. Describing $S(y)$ by a functional gives a method to check whether $x \in S(y)$. Nevertheless, one never obtains conclusions on elements for which $s(x,y) = 0$.

Since an s-transitive relation has certain transitive chains, a further idea is to use a multi-utility for elements x and y with $s(x,y) = 1$. The aim is to find a set U of functions $u : \mathcal{X} \to \mathbb{R}$ such that if $s(x,y) = 1$ it holds that

$$x \succsim y \quad \iff \quad u(x) \geq u(y),\ \forall u \in U.$$

The case $s(x,y) = 0$ is handled separately. Given such a representation, $x \succsim y \succsim z$ with $s(x,z) = 1$ implies $x \succsim z$ if $s(x,y) = s(y,z) = 1$. In the case that either $s(x,y)$ or $s(y,z)$ is zero, one cannot follow s-transitivity by only knowing the representation for elements for which s equals one. Hence, finding a representation for elements x and y with $s(x,y) = 0$ is crucial. The problem consists in representing $x \succsim y$ and $y \succsim z$ without the knowledge whether $s(x,z)$ is one or zero.

Remark 5.42. One could start by describing a representation of complete s-transitive relations. By Proposition 5.41, an s-transitive relation $\succsim$ is the intersection of its complete, s-transitive extensions. Provided any extension has a representation, we can collect them and thereby characterize $\succsim$.

Bibliography

[1] B. Acciaio and I. Penner. Dynamic convex risk measures. In G. Di Nunno and B. Öksendal (Eds.) Advanced Mathematical Methods for Finance, Springer:1–34, 2011.

[2] B. Acciaio, H. Föllmer, and I. Penner. Risk assessment for uncertain cash flows: Model ambiguity, discounting ambiguity, and the role of bubbles. *Finance and Stochastics*, 16(4):669–709, 2012.

[3] C.D. Aliprantis and K.C. Border. *Infinite Dimensional Analysis: a Hitchhiker's Guide*. Springer Verlag, 2006.

[4] P. Artzner, F. Delbaen, J.-M. Eber, and D. Heath. Coherent measures of risk. *Math. Finance*, 9(3):203–228, 1999.

[5] J.-P. Aubin and I. Ekeland. *Applied nonlinear analysis.* Dover Publications, Inc., Mineola, New York, 1984.

[6] A. Ben-Tal and M. Teboulle. Expected utility, penalty functions and duality in stochastic nonlinear programming. *Management Science*, 32:1445–1466, 1986.

[7] A. Ben-Tal and M. Teboulle. An old-new concept of convex risk measures: the optimized certainty equivalent. *Mathematical Finance*, 17(3):449–476, 2007.

[8] A.T. Bharucha-Reid. Fixed point theorems in probabilistic analysis. *Bulletin of the American Mathematical Society*, 82(5):641–657, 1976.

[9] S. Biagini and J. Bion-Nadal. Dynamic quasi-concave performance measures. *arXiv e-prints*, 2012.

[10] T.R. Bielecki, I. Cialenco, S. Drapeau, and M. Karliczek. Dynamic assessment indices. *arXiv e-prints*, 2013.

[11] T.R. Bielecki, I. Cialenco, I. Iyigunler, and R. Rodriguez. Dynamic conic finance: pricing and hedging in markets with transaction costs via dynamic coherent acceptability indices. *International Journal of Theoretical and Applied Finance*, 16 (1), 2013.

[12] T.R. Bielecki, I. Cialenco, and Z. Zhang. Dynamic coherent acceptability indices and their applications to finance. Forthcoming in *Math. Finance*, 2013.

[13] J. Bion-Nadal. Dynamic risk measures: time consistency and risk measures from BMO martingales. *Finance Stoch.*, 12(2):219–244, 2008.

[14] J. Bion-Nadal. Time consistent dynamic risk processes. *Stochastic Process. Appl.*, 119(2):633–654, 2009.

[15] K.C. Border. *Fixed point theorems with applications to economics and game theory*. Cambridge Univ. Press, 1989.

[16] H. Brézis and F.E. Browder. A general principle on ordered sets in nonlinear functional analysis. *Advances in Mathematics*, 21(355-364), 1976.

[17] J. Caristi. Fixed point theorems for mappings satisfying inwardness conditions. *Transactions of the American Mathematical Society*, 215:241–251, 1976.

[18] S. Cerreia-Vioglio, F. Maccheroni, M. Marinacci, and L. Montrucchio. Complete monotone quasiconcave duality. *Mathematics of Operations Research*, 36:321–339, 2011.

[19] S. Cerreia-Vioglio, F. Maccheroni, M. Marinacci, and L. Montrucchio. Risk measures: rationality and diversification. *Mathematical Finance*, 21:743–774, 2011.

[20] P. Cheridito and M. Kupper. Recursiveness of indifference prices and translation-invariant preferences. *Math. Financ. Econ.*, 2(3):173–188, 2009.

[21] P. Cheridito, F. Delbaen, and M. Kupper. Coherent and convex monetary risk measures for bounded càdlàg processes. *Stochastic Process. Appl.*, 112(1):1–22, 2004.

[22] P. Cheridito, F. Delbaen, and M. Kupper. Coherent and convex monetary risk measures for unbounded càdlàg processes. *Finance Stoch.*, 9(3):369–387, 2005.

[23] P. Cheridito, F. Delbaen, and M. Kupper. Dynamic monetary risk measures for bounded discrete-time processes. *Electron. J. Probab.*, 11:57–106, 2006.

[24] P. Cheridito, M. Kupper, and N. Vogelpoth. Conditional analysis on $\mathbb{R}^d$. *arXiv e-prints*, 2012.

[25] A.S. Cherny and D.B. Madan. New measures for performance evaluation. *Review of Financial Studies*, 22(7):2571–2606, 2009.

[26] A.S. Cherny and D.B. Madan. Markets as a counterparty: an introduction to conic finance. *International Journal of Theoretical and Applied Finance*, 13(8): 1149–1177, 2010.

[27] S. Dancs, M. Hegedüs, and P. Medvegyev. A general ordering and fixed point principle in complete metric spaces. *Acta Mathematica Scienta*, 46:381–388, 1983.

[28] K. Detlefsen and G. Scandolo. Conditional and dynamic convex risk measures. *Finance Stoch.*, 9(4):539–561, 2005.

[29] S. Drapeau. *Risk preferences and their robust representation.* PhD thesis, Humboldt-Universität zu Berlin, 2010.

[30] S. Drapeau and M. Kupper. Risk Preferences and their Robust Representation. *Mathematics of Operations Research*, 38(1):28–62, 2013.

[31] S. Drapeau, A. Jamneshan, M. Karliczek, and M. Kupper. The algebra of conditional sets and the concepts of conditional topology and compactness. *arXiv e-prints*, September 2013.

[32] S. Drapeau, M. Karliczek, M. Kupper, and M. Streckfuß. Brouwer fixed point theorem in $(L^0)^d$. *Fixed Point Theory and Applications*, 2013(1):301, 2013.

[33] S. Drapeau, M. Kupper, and A. Papapantoleon. A fourier approach to the computation of CV@R and optimized certainty equivalents. *Forthcoming in The Journal of Risk*, 2013.

[34] J. Dubra, F. Maccheroni, and E. A. Ok. Expected utility theory without the completeness axiom. *Journal of Economic Theory*, 115:118–133, 2004.

[35] B. Dushnik and E.W. Miller. Partially ordered sets. *American Journal of Mathematics*, 63(3):600–610, 1941.

[36] G. Eichfelder. Optimal elements in vector optimization with a variable ordering structure. *Journal of Optimization Theory and Applications*, 151(2):217–240, 2011.

[37] G. Eichfelder. Numerical procedures in multiobjective optimization with variable ordering structures. *Journal of Optimization Theory and Applications*, 2013.

[38] K.-T. Eisele and S. Taieb. Weak topologies for modules over rings of bounded random variables. *arXiv e-prints*, 2013.

[39] I. Ekeland. On the variational principle. *Journal of Mathematical Analysis and Applications*, 47:324–353, 1974.

[40] O. Evren and E. A. Ok. On the multi-utility representation of preference relations. *Journal of Mathematical Economics*, 47:554–563, 2011.

[41] M. Fabian, P. Habala, P. Hájek, V. Montesinos, and V. Zizler. *Banach space theory: the basis for linear and nonlinear analysis.* Springer, 2011.

[42] R. Fierro, C. Martinez, and C. H. Morales. Fixed point theorems for random lower semi-continuous mappings. *Fixed Point Theory and Applications*, 2009.

[43] D. Filipović, M. Kupper, and N. Vogelpoth. Separation and duality in locally L^0-convex modules. *Journal of Functional Analysis*, 256:3996–4029, 2009.

[44] P. C. Fishburn. Nontransitive measurable utility. *Journal of Mathematical Psychology*, 26:31–67, 1982.

[45] P. C. Fishburn. Interval graphs and interval orders. *Discrete Mathematics*, 55: 135–149, 1985.

[46] P. C. Fishburn. Nontransitive preferences in decision theory. *Journal of Risk and Uncertainty*, 4:113–134, 1991.

[47] P. C. Fishburn and W. T. Trotter. Geometric containment orders: a survey. *Order*, 15:167–182, 1999.

[48] H. Föllmer and A. Schied. Robust preferences and convex measures of risk. *Advances in Finance and Stochastics*, pages 39–56, 2002.

[49] H. Föllmer and A. Schied. *Stochastic finance: an introduction in discrete time*. de Gruyter Studies in Mathematics 27. Walter de Gruyter, Berlin, 2nd edition, 2004.

[50] M. Frittelli and M. Maggis. Conditional certainty equivalent. *Int. J. Theor. Appl. Finance*, 14(1):41–59, 2011.

[51] M. Frittelli and M. Maggis. Dual representation of quasi-convex conditional maps. *SIAM J. Financial Math.*, 2:357–382, 2011.

[52] M. Frittelli and E. Rosazza Gianin. Putting order in risk measures. *Journal of Banking and Finance*, 26:1473–1486, 2002.

[53] M. Frittelli and E. Rosazza Gianin. Dynamic convex risk measures. In G. Szegö, editor, *Risk Measures in the 21st Century*, pages 227–248. Wiley Finance Series, 2004.

[54] M. Frittelli and G. Scandolo. Risk measures and capital requirements for processes. *Math. Finance*, 16(4):589–612, 2006.

[55] A. Göpfert, C. Tammer, and C. Zălinescu. On the vectorial Ekeland's variational principle and minimal points in product spaces. *Nonlinear Analysis. Theory , Methods and Applications*, 39(909–922), 2000.

[56] A. Göpfert, H. Riahi, C. Tammer, and C. Zălinescu. *Variational methods in partially ordered spaces*, volume 17 of *CMS Books in Mathematics*. Springer Verlag, New York, 2003.

[57] T. Guo. Relations between some basic results derived from two kinds of topologies for a random locally convex module. *Journal of Functional Analysis*, 258(9):3024–3047, 2010.

[58] A. H. Hamel. Remarks to an equivalent formulation of Ekeland's variational principle. *Optimization*, 31:233–238, 1994.

[59] A. H. Hamel. Variational principle on metric and uniform spaces. *Habilitationsschrift, Martin-Luther Universität Halle-Wittenberg*, 2005.

[60] W.A. Kirk. Caristi's fixed point theorem and metric convexity. *Colloquium Mathematicae*, 38:81–86, 1976.

[61] M. Kupper and N. Vogelpoth. Complete L^0-normed modules and automatic continuity of monotone convex functions. *arXiv e-prints*, 2009.

[62] R. D. Luce. Semiorders and a theory of utility discrimination. *Econometrica*, 24 (2):178–191, 1956.

[63] J.D. Monk, S. Koppelberg, and R. Bonnet. *Handbook of boolean algebras.* North-Holland, 1989.

[64] A.B. Nemeth. Ekeland's variational principle in ordered abelian groups. *Nonlinear Analysis Forum*, 6(2):299–312, 2001.

[65] W. Oettli and M. Théra. Equivalents of Ekeland's Principle. *Bulletin of the Australian Mathematical Society*, 48((3)):385–392, 1993.

[66] J.-P. Penot and M. Volle. On quasi-convex duality. *Mathematics of Operations Research*, 15:597–625, 1990.

[67] N. Shahzad. Random fixed points of set-valued maps. *Nonlinear Analysis*, 45: 689–692, 2001.

[68] W. Takahashi. Existence theorems generalizing fixed point theorems for multivalued maps. *Fixed Point Theory and Applications*, volume 252 of *Pitman Research Notes in Mathematics*:397–406, 1991.

[69] C. Zălinescu. *Convex analysis in general vector spaces.* World Scientific, 2002.

Selbständigkeitserklärung

Ich erkläre, dass ich die vorliegende Arbeit selbständig und nur unter Verwendung der angegebenen Literatur und Hilfsmittel angefertigt habe.

Berlin, den 28.03.2014 Martin Karliczek